WHAT IS A PERIOD?

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The Riemann zeta function

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad , \qquad \Re(s) > 1$$

needs no introduction, probably : of great arithmetic interest, many of its properties remain unknown. It verifies

$$\zeta(2) = \frac{\pi^2}{6} \qquad \text{and} \qquad \zeta(4) = \frac{\pi^4}{90} \tag{1}$$

and, more generally, $\zeta(2n) \in \pi^{2n} \cdot \mathbb{Q}^\times$ for all positive n. On the other hand, the values at *odd* positive numbers are much more mysterious : only in 1978 was Roger Apéry able to prove that $\zeta(3) \notin \mathbb{Q}$ and, although it is widely believed that all numbers $\zeta(2n+1)$ should be transcendental, no proof of this transcendence has been provided so far.

We won't focus on these very interesting results. We will try to address another question : why is π appearing in (1) and at all positive even values? Is this a mere coincidence? Would it be possible, whatever that would mean, that

$$\zeta(2) \overset{?}{\in} e^2 \cdot \mathbb{Q}^\times \qquad \text{or} \qquad \zeta(4) \overset{?}{\in} \left(\frac{1}{\pi}\right)^4 \cdot \mathbb{Q}^\times \tag{2}$$

Well, no : it wouldn't be possible, for instance because it is false.

Very enlightening an answer, indeed.

In this informal talk, I will try to define what *periods* are (they are particular complex numbers, in fact) and I will discuss why it is reasonable to expect them to show off every now and then when discussing arithmetic-related questions. Then, a more enlightening answer to question (2) will turn out to be : π is a period, e and π^{-1} are, at least conjecturally, not. Therefore, it would not be possible for the zeta function to take rational multiples of these non-periods as values when evaluated at even positive integers.

Date: October 20$^{\text{th}}$, 2015.