Maximum flow problem

Network flows

- *Network*
  - Directed graph $G = (V, E)$
  - Source node $s \in V$, sink node $t \in V$
  - Edge capacities: $\text{cap} : E \rightarrow \mathbb{R}_{\geq 0}$
- *Flow:* $f : E \rightarrow \mathbb{R}_{\geq 0}$ satisfying
  1. Flow conservation constraints
  $$\sum_{e:\text{target}(e)=v} f(e) = \sum_{e:\text{source}(e)=v} f(e), \text{ for all } v \in V \setminus \{s,t\}$$
  2. Capacity constraints
  $$0 \leq f(e) \leq \text{cap}(e), \text{ for all } e \in E$$

Maximum flow problem

- *Excess:*
$$\text{excess}(v) = \sum_{e:\text{target}(e)=v} f(e) - \sum_{e:\text{source}(e)=v} f(e)$$
- If f is a flow, then $\text{excess}(v) = 0$, for all $v \in V \setminus \{s,t\}$
- *Value of a flow:* $\text{val}(f) = \text{excess}(t)$
- **Maximum flow problem:**
$$\max\{\text{val}(f) \mid f \text{ is a flow in } G\}$$
- Can be seen as a linear programming problem.

Lemma.
If f is a flow in G, then $\text{excess}(t) = -\text{excess}(s)$.

Maximum flow problem (2)

Proof. We have
$$\text{excess}(s) + \text{excess}(t) = \sum_{v \in V} \text{excess}(v) = 0.$$

- First "=": $\text{excess}(v) = 0$, for $v \in V \setminus \{s,t\}$
- Second "=": For any edge $e = (v, w)$, the flow through e appears twice in the sum, positively in $\text{excess}(w)$ and negatively in $\text{excess}(v)$.

Cuts

- A *cut* is a partition (S, T) of V, i.e., $T = V \setminus S$.

- (S,T) is an (s,t)-cut if $s \in S$ and $t \in T$.
- *Capacity* of (S,T)
$$\mathrm{cap}(S,T) = \sum_{E\cap(S\times T)} \mathrm{cap}(e)$$
- A cut is *saturated* by f if $f(e) = \mathrm{cap}(e)$, for all $e \in E\cap(S\times T)$, and $f(e) = 0$, for all $e \in E\cap(T\times S)$.

Lemma.
If f is a flow and (S,T) an (s,t)-cut, then

$$\mathrm{val}(f) \;=\; \sum_{e\in E\cap(S\times T)} f(e) \;-\; \sum_{e\in E\cap(T\times S)} f(e) \;\le\; \mathrm{cap}(S,T).$$

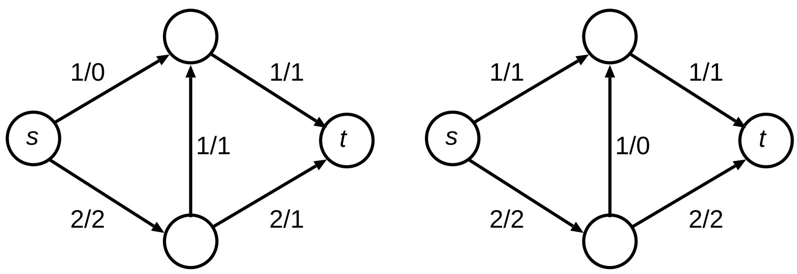
If S is saturated by f, then $\mathrm{val}(f) = \mathrm{cap}(S,T)$.

Maximum flow problem (2)

Proof. We have

$$\begin{aligned}
\mathrm{val}(f) &= -\mathrm{excess}(s) = -\sum_{u\in S}\mathrm{excess}(u) \\
&= \sum_{e\in E\cap(S\times T)} f(e) - \sum_{e\in E\cap(T\times S)} f(e) \\
&\le \sum_{e\in E\cap(S\times T)} \mathrm{cap}(e) \\
&= \mathrm{cap}(S)
\end{aligned}$$

For a saturated cut, the inequality is an equality.



Remarks.

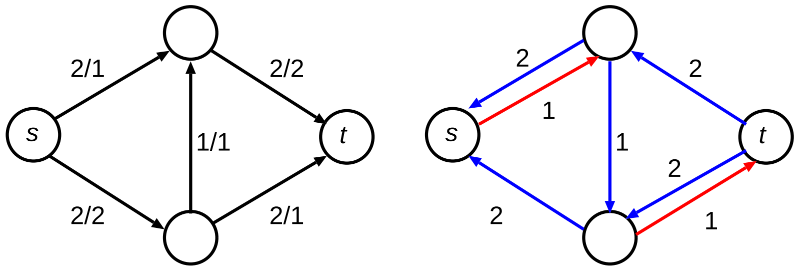
- A saturated cut proves the optimality of a flow.
- To show: for every maximal flow there is a saturated cut proving its optimality.

Residual network

The *residual network* G_f for a flow f in $G = (V,E)$ indicates the capacity unused by f. It is defined as follows:

- G_f has the same node set as G.
- For every edge $e = (v,w)$ in G, there are up to two edges e' and e'' in G_f:
    1. if $f(e) < \mathrm{cap}(e)$, there is an edge $e' = (v,w)$ in G_f with *residual capacity* $r(e') = \mathrm{cap}(e) - f(e)$.

2. if $f(e) > 0$, there is an edge $e'' = (w, v)$ in G_f with residual capacity $r(e'') = f(e)$.



Theorem.
Let f be an (s,t)-flow, let G_f be the residual graph w.r.t. f, and let S be the set of all nodes reachable from s in G_f.

- If $t \in S$, then f is not maximum.
- If $t \notin S$, then S is a saturated cut and f is maximum.

Proof (part 1).

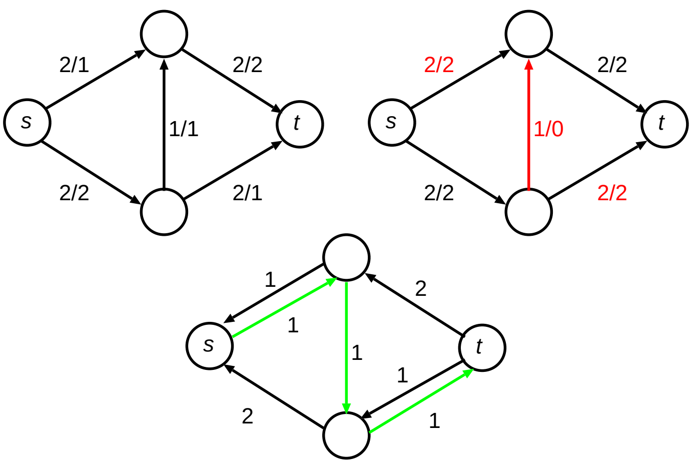
If t is reachable from s in G_f, then f is not maximal.

- Let p be a simple path from s to t in G_f.
- Let δ be the minimum residual capacity of an edge in p.
  By definition, $r(e) > 0$, for all edges e in G_f. Therefore, $\delta > 0$.
- Construct a flow f' of value $\mathrm{val}(f) + \delta$:

$$f'(e) = \begin{cases} f(e) + \delta, & \text{if } e' \in p \\ f(e) - \delta, & \text{if } e'' \in p \\ f(e), & \text{if neither } e' \text{ nor } e'' \text{ belongs to } p. \end{cases}$$

- f' is a flow and $\mathrm{val}(f') = \mathrm{val}(f) + \delta$.

Example.



Proof (part 2).

If t is not reachable from s in G_f, then f is maximal.

- Let S be the set of nodes reachable from s in G_f, and let $T = V \setminus S$.
- There is no edge (v, w) in G_f with $v \in S$ and $w \in T$.
- Hence
  - $f(e) = \text{cap}(e)$, for any $e \in E \cap (S \times T)$, and
  - $f(e) = 0$, for any $e \in E \cap (T \times S)$.
- Thus S is saturated and, by the Lemma, f is maximal.

Max-Flow-Min-Cut Theorem

Theorem.
The maximum value of a flow is equal to the minimum capacity of an (s, t)-cut:

$$\max\{\text{val}(f) \mid f \text{ is a flow}\} = \min\{\text{cap}(S, T) \mid (S, T) \text{ is an } (s, t)\text{-cut}\} \ .$$

Ford-Fulkerson Algorithm

1. Start with the zero flow, i.e., $f(e) = 0$, for all $e \in E$.
2. Construct the residual network G_f.
3. Check whether t is reachable from s.
   - if not, stop.
   - if yes, increase flow along an *augmenting path*, and iterate.

Analysis

- Let $|V| = n$ and $|E| = m$.
- Each iteration takes time $O(n + m)$.
- If capacities are arbitrary reals, the algorithm may run forever.

Integer capacities

- Suppose capacities are integers, bounded by C.
- v^* := value of maximum flow can be up to $(n - 1)C$.
- All flows constructed are integral (proof by induction).
- Every augmentation increases flow value by at least 1.
- Running time is $O((n + m)v^*) \longrightarrow$ *pseudo-polynomial*.

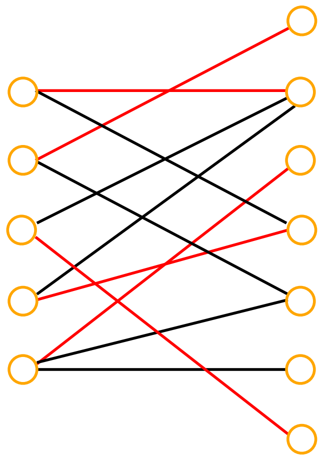
Edmonds-Karp Algorithm

- Compute *shortest* augmenting path, i.e., a shortest path from s to t in the residual network G_f, where each edge has distance 1.

- Apply, e. g., breadth-first search
- Resulting maximum flow algorithm can be implemented in $O(nm^2)$.

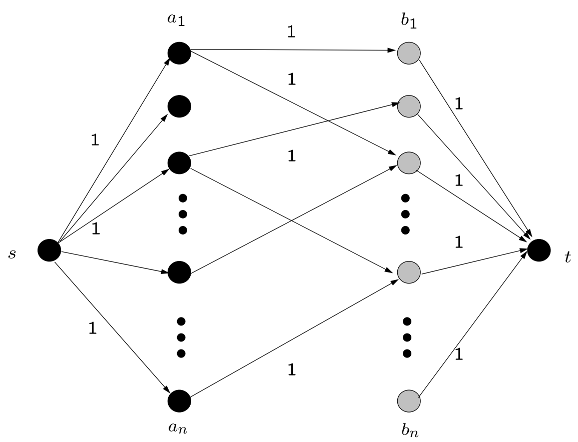
Bipartite matching

- $G = (V, E)$ undirected graph
- *Matching*: Subset of edges $M \subseteq E$, no two of which share an endpoint.
- *Maximum matching*: Matching of maximum cardinality.
- *Perfect matching*: Every vertex in V is matched.
- *G bipartite*: $V = A \cup B, A \cap B = \emptyset$, and each edge in E has one end in A and one end in B.



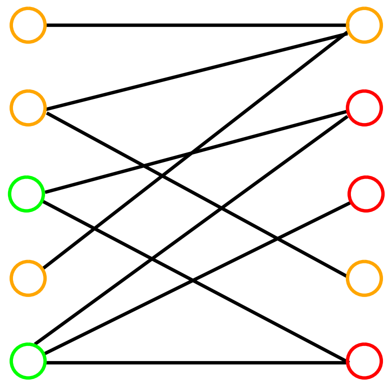
Reduction to a network flow problem

- Add a source s and edges (s, a) for $a \in A$, with capacity 1.
- Add a sink t and edges (b, t) for $b \in B$, with capacity 1.
- Direct edges in G from A to B, with capacity 1.
- Integral flows f correspond to matchings M, with $\mathrm{val}(f) = |M|$.
- Ford-Fulkerson takes time $O\big((m+n)n\big)$, since $v^* \leq n$.
- This can be improved to $O(\sqrt{n}\, m)$.



Marriage theorem

Theorem (Hall).
A bipartite graph $G = (A \cup B, E)$, with $|A| = |B| = n$, has a perfect matching if and only if for all $B' \subseteq B$, $|B'| \leq |N(B')|$, where $N(B')$ is the set of all neighbors of nodes in B'.

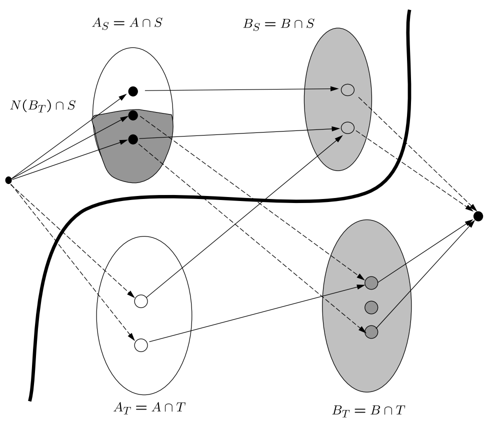


Proof

- Let (S,T) be an (s,t)-cut in the corresponding network.
- Let $A_S = A \cap S, A_T = A \cap T, B_S = B \cap S, B_T = B \cap T$.

$$\begin{aligned}
\text{cap}(S,T) &= \sum_{e \in E \cap S \times T} \text{cap}(e) \\
&= |A_T| + |B_S| + |N(B_T) \cap A_S| \\
&\geq |N(B_T) \cap A_T| + |N(B_T) \cap A_S| + |B_S| \\
&= |N(B_T)| + |B_S| \\
&\geq |B_T| + |B_S| = |B| = n
\end{aligned}$$

- By the max-flow min-cut theorem, the maximum flow is at least n.



König's theorem

- $G = (V,E)$ undirected graph

- $C \subseteq V$ is a *vertex cover* if every edge of G has at least one end in C.
- Lemma: For any matching M and any vertex cover C, we have $|M| \leq |C|$.
- **Theorem (König).** For a bipartite graph G,

$$\max\{|M| : M \text{ a matching }\} = \min\{|C| : C \text{ a vertex cover }\}.$$

Network connectivity

- $G = (V, E)$ directed graph, $s, t \in V, s \neq t$.
- **Theorem (Menger).** The maximum number of *arc-disjoint* paths from s to t equals the minimum number of arcs whose removal disconnects all paths from node s to node t.
- **Theorem (Menger).** The maximum number of *node-disjoint* paths from s to t equals the minimum number of nodes whose removal disconnects all paths from node s to node t.

Duality in linear programming

- Primal problem

$$z_P = \max\{\mathbf{c}^\mathbf{T} x \mid Ax \leq b, x \in \mathbb{R}^n\} \qquad \text{(P)}$$

- Dual problem

$$w_D = \min\{b^T u \mid A^T u = \mathbf{c}, u \geq 0\} \qquad \text{(D)}$$

General form

(P)			(D)		
min	$c^T x$		max	$u^T b$	
w.r.t.	$A_{i*}x \geq b_i$,	$i \in M_1$	w.r.t.	$u_i \geq 0$,	$i \in M_1$
	$A_{i*}x \leq b_i$,	$i \in M_2$		$u_i \leq 0$,	$i \in M_2$
	$A_{i*}x = b_i$,	$i \in M_3$		u_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$		$(A_{*j})^T u \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$		$(A_{*j})^T u \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$		$(A_{*j})^T u = c_j$,	$j \in N_3$

Duality theorems

- **Weak duality** If x^* is primal and u^* is dual feasible, then

$$c^T x^* \leq z_P \leq w_D \leq b^T u^*.$$

- **Strong duality** If both (P) and (D) have a finite optimum, then $z_P = w_D$.
- *Only four possibilities*
  1. z_P and w_D are both finite and equal.
  2. $z_P = +\infty$ and (D) is infeasible.
  3. $w_D = -\infty$ and (P) is infeasible.

4. (P) and (D) are both infeasible.

Maximum flow and duality

- Primal problem

$$\begin{aligned} \max \quad & \sum_{e:\text{source}(e)=s} x_e - \sum_{e:\text{target}(e)=s} x_e \\ \text{s.t.} \quad & \sum_{e:\text{target}(e)=v} x_e - \sum_{e:\text{source}(e)=v} x_e = 0, \quad \forall v \in V \setminus \{s,t\} \\ & 0 \le x_e \le c_e, \quad \forall e \in E \end{aligned}$$

- Dual problem

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e y_e \\ \text{s.t.} \quad & z_w - z_v + y_e \ge 0, \quad \forall e = (v,w) \in E \\ & z_s = 1, z_t = 0 \\ & y_e \ge 0, \quad \forall e \in E \end{aligned}$$

Maximum flow and duality (2)

- Let (y^*, z^*) be an optimal solution of the dual.
- Define $S = \{v \in V \mid z_v^* > 0\}$ and $T = V \setminus S$.
- (S,T) is a minimum cut.
- Max-flow min-cut theorem is a special case of linear programming duality.

Total unimodularity

- A matrix A is *totally unimodular* if each subdeterminant of A is $0, +1$ or -1.
- **Theorem (Hoffman and Kruskal).** $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b, x \ge 0\}$ is integral, i.e., $P = \text{conv}(P \cap \mathbb{Z}^n)$, for any $b \in \mathbb{Z}^m$.
- **Corollary.** $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff for any $b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$ both optima in the LP duality equation

$$\max\{c^T x \mid Ax \le b, x \ge 0\} = \{\min b^T u \mid A^T u \ge c, u \ge 0\}$$

  are attained by integral vectors (if they are finite).
- **Proposition.** The constraint matrix A arising in a maximum flow problem is totally unimodular.

References

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