## The Simplex algorithm (2)

Sticking to certain *pivoting* rules prevents cycling:

E.g., Bland's rule: among multiple candidates for entering/leaving the basis always choose the one with the smallest subscript.

This answers the third issue (Termination):

**Theorem.** The simplex method with Bland's rule terminates after a finite number of steps.

Proof. Since the algorithm does not cycle and there are only $\binom{n+m}{m}$ different dictionaries, the claim follows.

Unfortunately, pathological instances exist (e. g., the Klee-Minty cube), for which the Simplex method needs *exponential* time. However,

- in practice, the method is fast.
- other methods (e. g., Ellipsoid method) run in polynomial time.

## The Simplex algorithm (3)

We are left with only one issue (Initialization):

How do we find an initial dictionary if

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n \end{array}$$

has an infeasible origin?

Problems:

- Is there a feasible solution at all? (The problem might be infeasible)
- If so, how to find it?

## The Simplex algorithm (4)

Solution: Auxiliary problem

$$\begin{array}{lll} \min & x_0 & \text{(AUX)} \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 0, 1, \ldots, n \end{array}$$

Now, a feasible solution for (AUX) is easily found:

Set $x_j = 0$ for $j \in \{1, 2, \ldots, n\}$ and make $x_0$ sufficiently large.

Furthermore: the original problem has a feasible solution *if and only if* the optimum value of (AUX) is zero.

Thus, we solve (AUX) first.

## The Simplex algorithm (5)

*Example.*

$$\begin{array}{rl} \max & x_1 - x_2 + x_3 \\ \text{s. t.} & 2x_1 - x_2 + 2x_3 \leq 4 \\ & 2x_1 - 3x_2 + x_3 \leq -5 \\ & -x_1 + x_2 - 2x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{array} \quad \text{leads to} \quad \begin{array}{rl} \max & -x_0 \qquad \text{(AUX)} \\ \text{s. t.} & 2x_1 - x_2 + 2x_3 - x_0 \leq 4 \\ & 2x_1 - 3x_2 + x_3 - x_0 \leq -5 \\ & -x_1 + x_2 - 2x_3 - x_0 \leq -1 \\ & x_0, x_1, x_2, x_3 \geq 0 \end{array}$$

The first dictionary for (AUX) then looks like

$$\begin{aligned} x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\ x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\ x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0 \\ w &= -x_0 \ , \end{aligned}$$

which is also infeasible! So where's the advantage?

## The Simplex algorithm (6)

We can make it feasible by one single pivot, namely by having $x_0$ enter the basis and having $x_5$ leave it.

This yields the feasible dictionary

$$\begin{aligned} x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 &= 9 - 2x_2 - x_3 + x_5 \\ x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\ w &= -5 - 2x_1 + 3x_2 - x_3 - x_5 \ , \end{aligned}$$

from which we can read off the first feasible solution for (AUX)

$$x = (5, 0, 0, 0, 9, 0, 6) \quad \text{with} \quad w = -5$$

## The Simplex algorithm (7)

Two more iterations, namely

$$\begin{aligned} x_2 &= 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\ x_0 &= 2 - \frac{1}{4}x_1 - \frac{5}{4}x_3 + \frac{1}{4}x_5 + \frac{3}{4}x_6 \\ x_4 &= 7 - \frac{3}{2}x_1 - + \frac{5}{2}x_3 + \frac{1}{2}x_5 + \frac{1}{2}x_6 \\ w &= -2 + \frac{1}{4}x_1 + \frac{5}{4}x_3 - \frac{1}{4}x_5 - \frac{3}{4}x_6 \end{aligned} \quad \text{and} \quad \begin{aligned} x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \\ x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{3}{5}x_0 \\ x_4 &= 3 - x_1 - x_6 + 2x_0 \\ w &= -x_0 \end{aligned}$$

solve (AUX) and its optimal value is $w = 0$. Therefore, we can read off a first feasible solution

$$(0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0) \ \ldots$$

## The Simplex algorithm (8)

...and a first feasible dictionary:

$$
\begin{aligned}
x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{4}{5}x_5 + \frac{1}{5}x_6 \\
x_4 &= 3 - x_1 - x_6 \\
z &= x_1 - x_2 + x_3 = x_1 - (\frac{11}{5} + \frac{3}{5}x_1 + \frac{4}{5}x_5 + \frac{1}{5}x_6) + (\frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6) \\
&= -\frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6
\end{aligned}
$$

Now, we can go on with the regular Simplex method.

## The Simplex algorithm (9)

General method (*first phase* of two-phase Simplex):

We solve

$$
\begin{aligned}
\max \quad & -x_0 \qquad \text{(AUX)} \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i \qquad & i = 1, 2, \ldots, m \\
& x_j \geq 0 \qquad & j = 1, 2, \ldots, n
\end{aligned}
$$

by starting with an infeasible dictionary

$$
\begin{aligned}
x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij}x_j + x_0 \qquad & i = 1, 2, \ldots, m \\
w &= -x_0
\end{aligned}
$$

We arrive at a feasible dictionary by swapping $x_0$ with the “most infeasible” $x_{n+i}$, more precisely, with $x_{n+(\arg\min_{i=1,\ldots,m} b_i)}$.

## The Simplex algorithm (10)

One more special rule when solving (AUX):

Whenever $x_0$ is a candidate for leaving the basis, we pick it.

Why? Because we obtain a feasible solution with $x_0 = 0$ and thus $w = 0$ due to the properties of a dictionary.

Do other cases exist? After termination of phase one

- $x_0$ may be basic, and the value of $w$ is zero. But then, in the previous iteration, we had $w < 0$ and thus $x_0 > 0$ due to $w = -x_0$. So, we have not followed the special rule for picking $x_0$ whenever possible; thus, this case may not occur.
- $x_0$ may be basic, and the value of $w$ is non-zero. This case proves that the original problem is infeasible.

## The Simplex algorithm (11)

We are now ready for the

**Fundamental theorem of linear programming.** Every LP problem has the following three properties:

1. If it has no optimal solution, then it is either infeasible or unbounded.
2. If it has a feasible solution, then it has a basic feasible solution.
3. If it has an optimal solution, then it has a basic optimal solution.

Proof (constructive). The first phase of the two-phase Simplex algorithm either dicsovers that the problem is infeasible or computes a basic feasible solution. The second phase then finds a basic optimal solution or discovers that the problem is unbounded.

# Duality

## Duality: Introductory example

Consider

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\ \text{subject to} \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\ & 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Let us try to find a quick estimate on the optimal solution value $z^*$.

Lower bounds? Rather run Simplex...

Upper bounds?

## Duality: Introductory example (2)

Blackboard calculations lead to the *dual problem*

$$\begin{aligned} \min \quad & y_1 + 55y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + 5y_2 - y_3 \geq 4 \\ & -y_1 + y_2 + 2y_3 \geq 1 \\ & -y_1 + 3y_2 + 3y_3 \geq 5 \\ & 3y_1 + 8y_2 - 5y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

## Duality

In general, the dual of

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j & & \text{(primal problem)} \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & & i = 1, 2, \ldots, m \\ & x_j \geq 0 & & j = 1, 2, \ldots, n \end{aligned}$$

is

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} b_i y_i & & \text{(dual problem)} \\
\text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & & j = 1, 2, \ldots, n \\
& y_i \geq 0 & & i = 1, 2, \ldots, m
\end{aligned}
$$

**Lemma.** (Weak duality) $\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$ . Proof. Blackboard.