Linear programming

Optimization Problems

- *General optimization* problem

$$\max\{z(x) \mid f_j(x) \le 0, x \in D\} \text{ or } \min\{z(x) \mid f_j(x) \le 0, x \in D\}$$

  where $D \subseteq \mathbb{R}^n, f_j : D \to \mathbb{R}$, for $j = 1, \ldots, m$, $z : D \to \mathbb{R}$.

- *Linear optimization* problem

$$\max\{c^T x \mid Ax \underset{\geq}{\overset{\leq}{=}} b, x \in \mathbb{R}^n\}, \text{ with } c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

- *Integer* optimization problem: $x \in \mathbb{Z}^n$
- *0-1* optimization problem: $x \in \{0,1\}^n$

Feasible and optimal solutions

- Consider the optimization problem

$$\max\{z(x) \mid f_j(x) \le 0, x \in D, j = 1, \ldots, m\}$$

- A *feasible solution* is a vector $x' \in D \subseteq \mathbb{R}^n$ such that $f_j(x') \le 0$, for all $j = 1, \ldots, m$.
- The set of all feasible solutions is called the *feasible region*.
- An *optimal solution* x^* is a feasible solution such that

$$z(x^*) = \max\{z(x) \mid f_j(x) \le 0, x \in D, j = 1, \ldots, m\}.$$

- Feasible or optimal solutions
  - need not exist,
  - need not be unique.

Transformations

- $\min\{z(x) \mid x \in S\} = \max\{-z(x) \mid x \in S\}$.
- $f(x) \ge a$ if and only if $-f(x) \le -a$.
- $f(x) = a$ if and only if $f(x) \le a \wedge -f(x) \le -a$.

Lemma

Any linear programming problem can be brought to the form

$$\max\{c^T x \mid Ax \le b\} \quad \text{or} \quad \max\{c^T x \mid Ax = b, x \ge 0\}.$$

Proof: a) $a^T x \le \beta \rightsquigarrow a^T x + x' = \beta, x' \ge 0$ *(slack variable)*
b) x free $\rightsquigarrow x = x^+ - x^-$, $x^+, x^- \ge 0$.

Practical problem solving

1. Model building
2. Model solving
3. Model analysis

Example: Production problem

A firm produces n different goods using m different raw materials.

- b_i: availabe amount of the i-th raw material
- a_{ij}: number of units of the i-th material needed to produce one unit of the j-th good
- c_j: revenue for one unit of the j-th good.

Decide how much of each good to produce in order to maximize the total revenue $\rightsquigarrow$ *decision variables* x_j.

Linear programming formulation

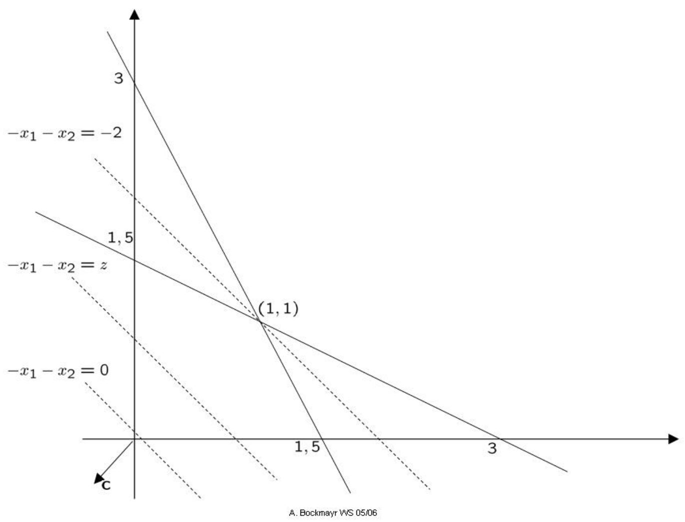
$$
\begin{array}{lccccccc}
\max & c_1x_1 & + & \cdots & + & c_nx_n & & \\
\text{w.r.t.} & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & \leq & b_1, \\
 & \vdots & & & & \vdots & & \\
 & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & \leq & b_m, \\
 & x_1, & & \ldots & & ,x_n & \geq & 0.
\end{array}
$$

In matrix notation:

$$\max\{c^Tx \mid Ax \leq b, x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, x \in \mathbb{R}^n$.

Geometric illustration



$$
\begin{array}{lccccc}
\max & x_1 & + & x_2 & & \\
\text{w.r.t.} & x_1 & + & 2x_2 & \leq & 3 \\
 & 2x_1 & + & x_2 & \leq & 3 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}
$$

Polyhedra

- *Hyperplane* $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$, $a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$
- *Closed halfspace* $\overline{H} = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$
- *Polyhedron* $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- *Polytope* $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ l \leq x \leq u\}$

The feasible set

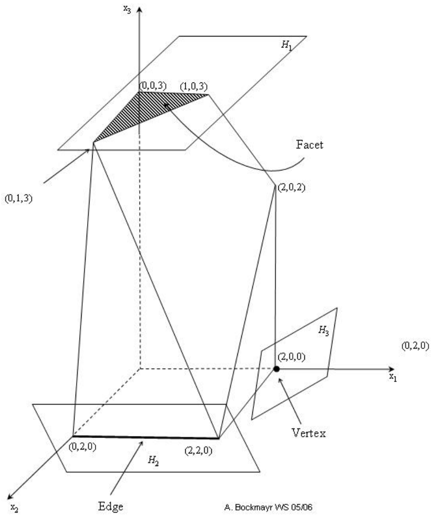
$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

of a linear optimization problem is a polyhedron.

Vertices, Faces, Facets

- $P \subseteq \overline{H}, H \cap P \neq \emptyset$ *(Supporting hyperplane)*
- $F = P \cap H$ *(Face)*
- $\dim(F) = 0$ *(Vertex)*
- $\dim(F) = 1$ *(Edge)*
- $\dim(F) = \dim(P) - 1$ *(Facet)*
- P *pointed:* P has at least one vertex.

Illustration



Simplex Algorithm: Geometric view

Linear optimization problem

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{LP}$$

Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of P.
2. Proceed from vertex to vertex along edges of P such that the objective function $z = c^T x$ increases.
3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which z is unbounded.