Combinatorial Optimization and Integer Linear Programming

Combinatorial Optimization: Introduction

Many problems arising in practical applications have a special, *discrete* and *finite*, nature:

Definition. (Linear Combinatorial Optimization Problem)
Given

- a finite set E (the ground set),
- a subset $\mathcal{F} \subseteq 2^E$ (the set of feasible solutions),
- a cost function $c : E \longrightarrow \mathbb{R}$,

find a set $F^* \in \mathcal{F}$ such that

$$c(F^*) := \sum_{e \in F^*} c(e)$$

is maximal or minimal.

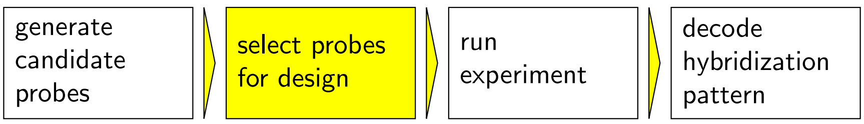
Examples: Shortest Path, Traveling Salesman, and many many more. . .
Just in bioinformatics: Alignments, Threading, Clone-Probe Mapping, Probe Selection, *De Novo* Peptide Sequencing, Side-Chain Placement, Maximum-weight Connected Subgraph in PPI Networks, Genome Rearrangements, Cluster Editing, Finding Regulatory Modules, Finding Approximate Gene Clusters, and many more. . .

Combinatorial Optimization: Introduction (2)

Example. Optimal Microarray Probe Selection

Experimental setup (group testing):

- Goal: determine presence of *targets* in sample
- *probes* hybridize with *targets* $\longrightarrow$ hybridization pattern



Selection phase:

- *unique* probes are easy to decode but difficult to find (similarities, errors, add. constraints, . . .)
- $\longrightarrow$ consider *non-unique probes*
- Task: choose *few* probes that still allow to infer which targets are in the sample

Combinatorial Optimization: Introduction (3)

Example hybridization matrix $(H)_{ij}$:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
t_1	1	1	1	0	1	1	0	0	0
t_2	1	0	1	1	0	0	1	1	0
t_3	0	1	1	1	0	1	1	0	1
t_4	0	1	0	0	1	0	1	1	1

Assume: no errors, only *one* target present in sample

Combinatorial Optimization: Introduction (4)

Example hybridization matrix $(H)_{ij}$:

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
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Assume: no errors, only *one* target present in sample

Combinatorial Optimization: Introduction (5)

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Assume: no errors, *two* targets present, e.g., t_2 and t_3

Combinatorial Optimization: Introduction (6)

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	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
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t_4	0	1	0	0	1	0	1	1	1

Assume: no errors, *two* targets present, e.g., t_2 and t_3

Combinatorial Optimization: Introduction (7)

We want to solve the following problem.

Definition. Probe Selection Problem (PSP)

- Given an incidence matrix H, $d \in \mathbb{N}$, and $c \in \mathbb{N}$,
- find the smallest subset $D \subseteq N$, such that
  - all *targets* are covered by at least d probes
  - all different subsets of targets S and T up to cardinality c are d-separable with respect to D

Observation. PSP is a combinatorial optimization problem, because

- ground set = candidate probes, i.e., $E := \{1, 2, \dots, n\}$.
- feasible solutions = feasible designs, i.e.,

$$\mathcal{F} := \{D \in 2^E \mid D \text{ satisfies coverage and separation constraints}\}$$

- all costs $c(e) := 1$.

Combinatorial Optimization: Introduction (8)

More examples. What about

$$\min\{3x^2 + 2 \mid x \in \mathbb{R}\} \text{ ?}$$

Or

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 3 \\ & 3x_1 - x_2 \leq 5 \\ & x_1, x_2 \in \mathbb{N} \text{ ?} \end{aligned}$$

Interesting combinatorial problems have an exponential number of feasible solutions. [Otherwise, a straightforward polynomial-time algorithm finds optimal solutions.]

Combinatorial optimization: find solutions faster than by complete enumeration.

Combinatorial Optimization (9)

Now, given a combinatorial optimization problem $C = (E, \mathcal{F}, c)$, we define, for each feasible solution $F \in \mathcal{F}$, its *characteristic vector* $\chi^F \in \{0,1\}^E$ as

$$\chi_e^F := \begin{cases} 1 & e \in F \\ 0 & \text{otherwise} \end{cases} .$$

Then, assuming the objective is to maximize, C can be seen as maximizing over a polytope, i. e.,

$$\max\{c^T x \mid x \in \operatorname{conv}\{\chi^F \in \{0,1\}^E \mid F \in \mathcal{F}\}\ .$$

Why *polytope*?

Theorem. (Minkowski 1896, Weyl 1935)
Each polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b, l \leq x \leq u\}$ can be written as

$$P = \operatorname{conv}(V)$$

where V is a finite subset of $\mathbb{R}^n$ *and vice versa*.

Combinatorial Optimization (10)

It is possible to switch between these descriptions as $\mathcal{H}$*-polytope* (halfspaces) and $\mathcal{V}$*-polytope* (vertices) with the *Fourier-Motzkin elimination* method.

Example.

Consider the $\mathcal{V}$-polytope defined by

$$P = \operatorname{conv}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

End (Example).

So, we can just compute the $\mathcal{H}$-polytope $\{x \in \mathbb{R}^n \mid Ax \leq b\}$ for C and optimize over it using, e. g., the Simplex method?

Unfortunately, it is not so easy:

- In general, we cannot find A and b in polynomial time.
- The size of A and b might be exponential.
- The coefficients in A and b can be exponentially large.

Combinatorial Optimization (11)

A little bit of light... often, finding an *integer linear programming (ILP) formulation* is easier:

$$\max\{c'^T x' \mid A'x' \leq b, x' \in \mathbb{Z}\}\ .$$

But: solving LPs is easy, solving ILPs is not!