Solving the MWT

Recall the ILP for the MWT. We can obtain a solution to the MWT problem by solving the following ILP:

$$
\begin{array}{rll}
\max & \sum_{e_i \in E} \omega_i x_i & \\
\text{subject to} & \sum_{e_i \in C \cap E} x_i \leq |C \cap E| - 1 & \text{for all critical mixed cycles } C \\
& x_i \in \{0,1\} & \text{for all } i = 1, \dots, n
\end{array}
$$

We showed before that this ILP describes the solution to the Maximum Weight Trace problem. The first step is to have a closer look at the MWT-polytope.

Solving the MWT (2)

Let $\mathcal{T} := \{T \subseteq E \mid T \text{ is a trace}\}$ be the set of all feasible solutions. We define the *MWT polytope* as the convex hull of all incidence vectors of E that are feasible, i. e.,

$$P_{\mathcal{T}}(G) := \operatorname{conv}\{\chi^T \in \{0,1\}^{|E|} \mid T \in \mathcal{T}\} \ ,$$

where the *incidence vector* χ^T for a subset $T \subseteq E$ is defined by setting $\chi^T_e = 1$ if $e \in E$ and setting $\chi^T_e = 0$ if $e \notin E$.

We have a closer look at the facial structure of the polytope, that means we try to *identify facet-defining classes of inequalities*. The following theorem is our main tool.

Identifying facet-defining classes of polytope

Theorem. Let $P \subseteq \mathbb{Q}^d$ be a full dimensional polyhedron. If F is a (nonempty) face of P then the following assertions are equivalent.

1. F is a facet of P.
2. $\dim(F) = \dim(P) - 1$, where $\dim(P)$ is the maximum number of affinely independent points in P minus one.
3. There exists a valid inequality $c^T x \leq c_0$ with respect to P with the following three properties:
   (a) $F = \{x \in P \mid c^T x = c_0\}$
   (b) There exists a vector $\hat{x} \in P$ such that $c^T \hat{x} < c_0$.
   (c) If $a^T x \leq a_0$ is a valid inequality for P such that $F \subseteq \bar{F} = \{x \in P \mid a^T x = a_0\}$ then there exists a number $\lambda \in \mathbb{Q}$ such that $a^T = \lambda \cdot c^T$ and $a_0 = \lambda \cdot c_0$.

Identifying facet-defining classes of polytope (2)

Assertions 2 and 3 provide the two basic methods to prove that a given inequality $c^T x \leq c_0$ is facet-defining for a polyhedron P.

The first method (Assertion 2), called the *direct* method, consists of exhibiting a set of $d = \dim(P)$ vectors $x_1, \dots, x_d$ satisfying $c^T x_i = c_0$ and showing that these vectors are affinely independent.

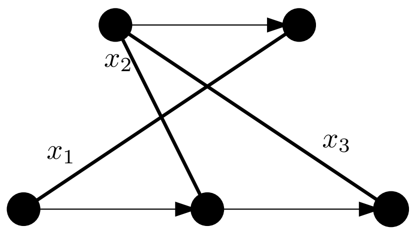
The *indirect* method (Assertion 3) is the following: We assume that

$$\{x \mid c^T x = c_0\} \subseteq \{x \mid a^T x = a_0\}$$

for some valid inequality $a^T x \leq a_0$ and prove that there exists a $\lambda > 0$ such that $a^T = \lambda \cdot c^T$ and $a_0 = \lambda \cdot c_0$.

Clique inequalities

Now we describe a class of valid, facet-defining inequalities for the MWT problem, focusing first on the *pairwise case*. In the case of two sequences, consider the following extended alignment graph:



This gives rise to the following set of inequalities:

$$x_1 + x_2 \leq 1,\ x_1 + x_3 \leq 1,\ x_2 + x_3 \leq 1$$

However, it is clear that only one of the three edges can be realized by an alignment. Hence, inequality $x_1 + x_2 + x_3 \leq 1$ is *valid* and more stringent. Indeed it cuts off the fractional solution $x_1 = x_2 = x_3 = \frac{1}{2}$.

Clique inequalities (2)

If $C \subseteq E$ is a set of alignment edges such that each pair forms a mixed cycle, it is called a *clique* (since it forms a clique in the *conflict graph*).

The conflict graph of a combinatorial optimization problem has a node for each object and an edge between pairs of conflicting objects). In general the *clique* inequalities

$$\sum_{e \in C} x_e \leq 1$$

are valid for the MWT problem.

Are they also facet-defining for the MWT polytope?

Theorem.
Let $C \subseteq E$ be a *maximal* clique. Then the inequality $\sum_{e \in C} x_e \leq 1$ is facet-defining for $P_{\mathcal{T}}(G)$.

Clique inequalities (3)

Proof.
We choose the direct way, which means we have to find n affinely independent vectors satisfying $\sum_{e \in C} x_e = 1$. This can be easily achieved. Assume without loss of generality that $|E \setminus C| \neq \emptyset$. We first construct $|C|$ many solutions by choosing a single edge in C.

Then for each edge $e \notin C$ there must be an edge $f \in C$ which does not form a mixed cycle with e (otherwise C is not maximal). Hence we can construct as set of solutions $\{e, f\}$, $\forall e \notin C$. This means we have for all n edges a solution satisfying the clique inequality with equality, and they are clearly affinely independent.

Clique inequalities (4)

But how do we efficiently find violated clique inequalities? How do we solve the separation problem? We define the following relation on edges:

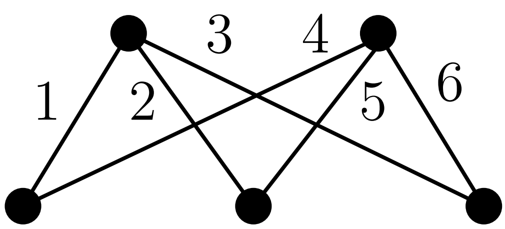
Definition.
Let $K_{p,q}$ be the complete bipartite graph with nodes $x_1, \dots, x_p$ and $y_1, \dots, y_q$. Define the strict partial order '$\prec$' on the edges of $K_{p,q}$ as follows:

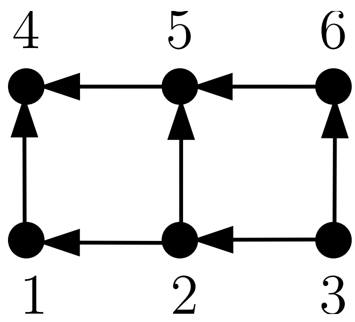
$$e = (x_i, y_j) \prec f = (x_k, y_l) \text{ iff}$$
$$(i > k \text{ and } j \leq l) \text{ or } (i = k \text{ and } j < l).$$

Observe that for two sequences the alignment graph (V, E) is a subgraph of $K_{p,q}$ and that two edges e and f form a mixed cycle in the input graph iff either $e \prec f$ or $f \prec e$.

Clique inequalities (5)

Definition.
Let $PG(K_{p,q})$ be the $p \times q$ directed grid graph with arcs going from right to left and from bottom to top. Row r, $1 \leq r \leq p$ of $PG(K_{p,q})$ contains q nodes which correspond from left to right to the q edges that go between node x_{p-r+1} and node $y_1, \dots, y_q$ in $K_{p,q}$. We call $PG(K_{p,q})$ the *pairgraph* of $K_{p,q}$ and we call a node of the pairgraph *essential* if it corresponds to an edge in E.





Clique inequalities (6)

The graph $PG(K_{p,q})$ has exactly one source and one sink and there is a path from node n_2 to node n_1 in $PG(K_{p,q})$ iff $e_1 \prec e_2$ for the corresponding edges e_1, e_2 in $K_{p,q}$.

Lemma.
Let $P = n_1, \dots, n_{p+q}$ be a source-to-sink path in $PG(K_{p,q})$ and let $e_1, \dots, e_l$, $l \leq p+q$, be the edges in E that correspond to essential nodes in P. Then $e_1, \dots, e_l$ is a clique of the input extended alignment graph if $l \geq 2$. Moreover, every maximal clique in the input extended alignment graph can be obtained in this way.

Clique inequalities (7)

Proof.
For any two nodes n_i and n_j in $PG(K_{p,q})$ with $i > j$ the corresponding edges e_i and e_j are in relation $e_i \prec e_j$ and hence form a mixed cycle in G. Thus $\{e_1, \dots, e_l\}$ is a clique of G. Conversely, the set of edges in any clique C of G is linearly ordered by $\prec$ and hence all maximal cliques are induced by source-to-sink paths in $PG(K_{p,q})$.

Clique inequalities (8)

We can now very easily use the pairgraphs for each pair of sequences to separate the clique inequalities.

Assume the solution $\bar{x}$ of the current LP-relaxation is fractional. Our problem is to find a clique C which violates the clique inequality

$$\sum_{e \in C \cap E} \bar{x}_e \leq 1 \ .$$

Assign the cost $\bar{x}_e$ to each essential node v_e in $PG(K_{p,q})$ (essential nodes are the nodes that correspond to the edges in E) and 0 to non-essential nodes.

Clique inequalities (9)

Then compute the longest source-to-sink path C in $PG(K_{p,q})$. If the cost of C is greater than 1, i.e.,

$$\sum_{e \in C \cap E} \overline{x}_e > 1$$

we have found a violated clique inequality.

Since $PG(K_{p,q})$ is acyclic, such a path can be found in polynomial time.

[Caution: We will not go deeper into this, but it is necessary to make a sparse version of the PG in the case of a non-complete bipartite graph. This has to be done such that its size ist still polynomial and each path encodes a maximal clique. Nevertheless, the trick with essential and non-essential nodes will work and leads to correct separation results.]

Mixed cycle inequalities (10)

Now we describe how to solve the separation problem for the mixed-cycle inequalities. Assume the solution $\overline{x}$ of the linear program is fractional.

First assign the cost $1 - \overline{x}_e$ to each edge $e \in E$ and 0 to all $a \in H$. Then we compute for each node $s_{i,j}$, $1 \leq i \leq k$, $1 \leq j < n_i$ the shortest path from $s_{i,j+1}$ to $s_{i,j}$. If there is such a shortest path P, and its cost is less than 1, i.e.,

$$\sum_{e \in P} (1 - \overline{x}_e) < 1 \ ,$$

we have found a violated inequality, namely

$$\sum_{e \in P} \overline{x}_e > |P| - 1 \ ,$$

since P together with the arc $(s_{i,j}, s_{i,j+1})$ forms a mixed cycle.