Duality (2)

Theorem. (Duality Theorem)
If the primal has an optimal solution x^* then the dual has an optimal solution y^* such that

$$c^T x^* = {y^*}^T b$$

Proof. (Constructive! $\longrightarrow$ blackboard)

Some properties:

- The dual of the dual is the primal (exercise).
- Corollary: Primal has an optimal solution *iff* dual has an optimal solution.
- Corollary: Primal unbounded $\longrightarrow$ dual infeasible.
- Corollary: Dual unbounded $\longrightarrow$ primal infeasible.
- Primal and dual can both be infeasible (Example).

Duality (3)

Practical implications:

- It might be faster to run the Simplex algorithm on the dual.
- Duality provides an elegant way of *proving optimality*. So, solutions come with a certificate, which is always good.

Simplex Algorithm: Geometric view

- Hyperplane $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}, a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$
- Closed halfspace $\overline{H} = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$
- Polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- Polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ l \leq x \leq u\}$

The feasible set

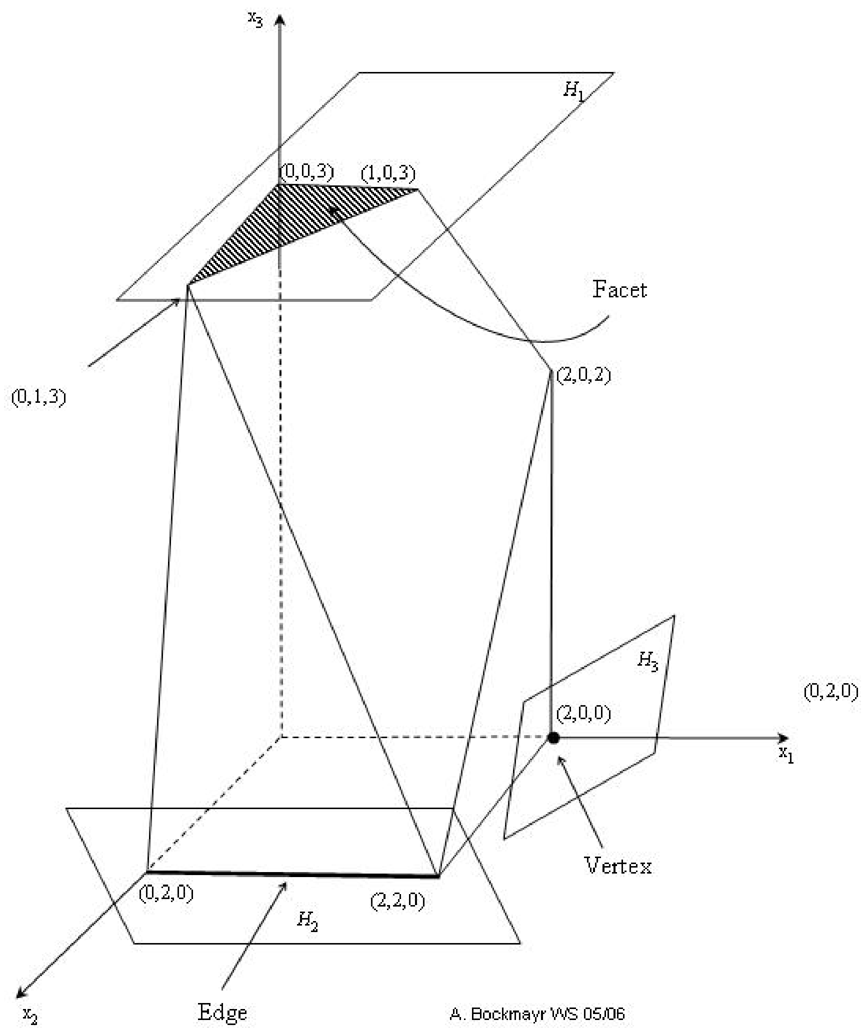
$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

of a linear optimization problem is a polyhedron.

Simplex Algorithm: Geometric view (2)

- $P \subseteq \overline{H}, H \cap P \neq \emptyset$ (Supporting hyperplane)
- $F = P \cap H$ (Face)
- $\dim(F) = 0$ (Vertex)
- $\dim(F) = 1$ (Edge)
- $\dim(F) = \dim(P) - 1$ (Facet)

Simplex Algorithm: Geometric view (3)



Simplex Algorithm: Geometric view (4)

Linear optimization problem

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n, x \geq 0\} \tag{LP}$$

Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of P.

2. Proceed from vertex to vertex along edges of P such that the objective function $z = c^T x$ increases.

3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which z is unbounded.