The Simplex algorithm (2)

Sticking to certain *pivoting* rules prevents cycling:

E.g., Bland's rule: among multiple candidates for entering/leaving the basis always choose the one with the smallest subscript.

This answers the third issue (Termination):

Theorem. The simplex method with Bland's rule terminates after a finite number of steps.

Proof. Since the algorithm does not cycle and there are only $\binom{n+m}{m}$ different dictionaries, the claim follows.

Unfortunately, pathological instances exist (e. g., the Klee-Minty cube), for which the Simplex method needs *exponential* time. However,

- in practice, the method is fast.
- other methods (e. g., Ellipsoid method) run in polynomial time.

The Simplex algorithm (3)

We are left with only one issue (Initialization):

How do we find an initial dictionary if

$$
\begin{array}{rll}
\max & \displaystyle\sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \le b_i & i = 1, 2, \ldots, m \\
 & x_j \ge 0 & j = 1, 2, \ldots, n
\end{array}
$$

has an infeasible origin?

Problems:

- Is there a feasible solution at all? (The problem might be infeasible)
- If so, how to find it?

The Simplex algorithm (4)

Solution: Auxiliary problem

$$
\begin{aligned}
\text{min} \quad & x_0 & & \text{(AUX)} \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i & & i = 1, 2, \ldots, m \\
& x_j \geq 0 & & j = 0, 1, \ldots, n
\end{aligned}
$$

Now, a feasible solution for (AUX) is easily found:

Set $x_j = 0$ for $j \in \{1, 2, \ldots, n\}$ and make x_0 sufficiently large.

Furthermore: the original problem has a feasible solution *if and only if* the optimum value of (AUX) is zero.

Thus, we solve (AUX) first.

The Simplex algorithm (5)

Example.

$$\begin{array}{ll} \max & x_1 - x_2 + x_3 \\ \text{s. t.} & 2x_1 - x_2 + 2x_3 \leq 4 \\ & 2x_1 - 3x_2 + x_3 \leq -5 \\ & -x_1 + x_2 - 2x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

leads to

$$\begin{array}{ll} \max & -x_0 \qquad \text{(AUX)} \\ \text{s. t.} & 2x_1 - x_2 + 2x_3 - x_0 \leq 4 \\ & 2x_1 - 3x_2 + x_3 - x_0 \leq -5 \\ & -x_1 + x_2 - 2x_3 - x_0 \leq -1 \\ & x_0, x_1, x_2, x_3 \geq 0 \end{array}$$

The first dictionary for (AUX) then looks like

$$\begin{aligned} x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\ x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\ x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0 \\ w &= -x_0 \ , \end{aligned}$$

which is also infeasible! So where's the advantage?

The Simplex algorithm (6)

We can make it feasible by one single pivot, namely by having x_0 enter the basis and having x_5 leave it.

This yields the feasible dictionary

$$\begin{aligned} x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 &= 9 - 2x_2 - x_3 + x_5 \\ x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \\ w &= -5 - 2x_1 + 3x_2 - x_3 - x_5 \ , \end{aligned}$$

from which we can read off the first feasible solution for (AUX)

$$x = (5, 0, 0, 0, 9, 0, 6) \quad \text{with} \quad w = -5$$

The Simplex algorithm (7)

Two more iterations, namely

$$\begin{aligned}
x_2 &= 1 + \frac{3}{4}x_1 + \frac{3}{4}x_3 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\
x_0 &= 2 - \frac{1}{4}x_1 - \frac{5}{4}x_3 + \frac{1}{4}x_5 + \frac{3}{4}x_6 \\
x_4 &= 7 - \frac{3}{2}x_1 - +\frac{5}{2}x_3 + \frac{1}{2}x_5 + \frac{1}{2}x_6 \\
w &= -2 + \frac{1}{4}x_1 + \frac{5}{4}x_3 - \frac{1}{4}x_5 - \frac{3}{4}x_6
\end{aligned}$$

and

$$\begin{aligned}
x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{4}{5}x_0 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{3}{5}x_0 \\
x_4 &= 3 - x_1 - x_6 + 2x_0 \\
w &= -x_0
\end{aligned}$$

solve (AUX) and its optimal value is $w = 0$. Therefore, we can read off a first feasible solution

$$(0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0) \quad \ldots$$

The Simplex algorithm (8)

. . . and a first feasible dictionary:

$$
\begin{aligned}
x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 \\
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{4}{5}x_5 + \frac{1}{5}x_6 \\
x_4 &= 3 - x_1 - x_6 \\
z &= x_1 - x_2 + x_3 = x_1 - (\frac{11}{5} + \frac{3}{5}x_1 + \frac{4}{5}x_5 + \frac{1}{5}x_6) + (\frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6) \\
&= -\frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6
\end{aligned}
$$

Now, we can go on with the regular Simplex method.

The Simplex algorithm (9)

General method (*first phase* of two-phase Simplex):

We solve

$$\begin{array}{lll} \max & -x_0 \quad \text{(AUX)} & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij}x_j - x_0 \leq b_i & i = 1, 2, \ldots, m \\ & x_j \geq 0 & j = 1, 2, \ldots, n \end{array}$$

by starting with an infeasible dictionary

$$\begin{array}{ll} x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j + x_0 & i = 1, 2, \ldots, m \\ w = -x_0 & \end{array}$$

We arrive at a feasible dictionary by swapping x_0 with the "most infeasible" x_{n+i}, more precisely, with $x_{n+(\arg\min_{i=1,\ldots,m} b_i)}$.

The Simplex algorithm (10)

One more special rule when solving (AUX):

Whenever x_0 is a candidate for leaving the basis, we pick it.

Why? Because we obtain a feasible solution with $x_0 = 0$ and thus $w = 0$ due to the properties of a dictionary.

Do other cases exist? After termination of phase one

- x_0 may be basic, and the value of w is zero. But then, in the previous iteration, we had $w < 0$ and thus $x_0 > 0$ due to $w = -x_0$. So, we have not followed the special rule for picking x_0 whenever possible; thus, this case may not occur.
- x_0 may be basic, and the value of w is non-zero. This case proves that the original problem is infeasible.

The Simplex algorithm (11)

We are now ready for the

Fundamental theorem of linear programming. Every LP problem has the following three properties:

1. If it has no optimal solution, then it is either infeasible or unbounded.

2. If it has a feasible solution, then it has a basic feasible solution.

3. If it has an optimal solution, then it has a basic optimal solution.

Proof (constructive). The first phase of the two-phase Simplex algorithm either dicsovers that the problem is infeasible or computes a basic feasible solution. The second phase then finds a basic optimal solution or discovers that the problem is unbounded.

Duality

Duality: Introductory example

Consider

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\ \text{subject to} \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\ & 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Let us try to find a quick estimate on the optimal solution value z^*.

Lower bounds? Rather run Simplex...

Upper bounds?

Duality: Introductory example (2)

Blackboard calculations lead to the *dual problem*

$$
\begin{array}{rl}
\min & y_1 + 55y_2 + 3y_3 \\
\text{subject to} & y_1 + 5y_2 - y_3 \geq 4 \\
 & -y_1 + y_2 + 2y_3 \geq 1 \\
 & -y_1 + 3y_2 + 3y_3 \geq 5 \\
 & 3y_1 + 8y_2 - 5y_3 \geq 3 \\
 & y_1, y_2, y_3 \geq 0
\end{array}
$$

Duality

In general, the dual of

$$\max \sum_{j=1}^{n} c_j x_j \qquad \text{(primal problem)}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

is

$$\min \sum_{i=1}^{m} b_i y_i \qquad \text{(dual problem)}$$

$$\text{subject to} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad j = 1, 2, \ldots, n$$

$$y_i \geq 0 \qquad i = 1, 2, \ldots, m$$

Lemma. (Weak duality) $\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$. Proof. Blackboard.