Constraint Programming

Constraint Programming

- Basic idea: Programming with constraints, i.e. constraint solving embedded in a programming language
- Constraints: linear, non-linear, finite domain, Boolean, . . .
- Programming: logic, functional, object-oriented, imperative, concurrent, . . .
- Systems: Prolog III/IV, CHIP, ECLIPSE, ILOG, OCRE, NCL, . . .

Finite Domain Constraints

Constraint satisfaction problem (CSP)

- n variables $x_1, \ldots, x_n$
- For each variable x_j a finite domain D_j of possible values, often $D_j \subset \mathbb{N}$.
- m constraints $C_1, \ldots, C_m$, where $C_i \subseteq D_{i_1} \times \ldots \times D_{i_{k_i}}$ is a relation between k_i variables $x_{i_1}, \ldots, x_{i_{k_i}}$. Write also $C_{i_1, \ldots, i_{k_i}}$.
- A solution is an assignment of a value D_j to x_j, for each $j = 1, \ldots, n$, such that all relations C_i are satisfied.

Coloring Problem

- Decide whether a map can be colored by 3 colors such that neighboring regions get different colors.

- For each region a variable x_j with domain $D_j = \{\text{red, green, blue}\}$.

- For each pair of variables x_i, x_j corresponding to two neighboring regions, a constraint $x_i \neq x_j$.

- NP-complete problem.

Resolution by Backtracking

- Instantiate the variables in some order.
- As soon as all variables in a constraint are instantiated, determine its truth value.
- If the constraint is not satisfied, backtrack to the last variable whose domain contains unassigned values, otherwise continue instantiation.

Efficiency Problems

Mackworth 77

1. If the domain D_j of a variable x_j contains a value v that does not satisfy C_j, this will be the cause of repeated instantiation followed by immediate failure.

2. If we instantiate the variables in the order $x_1, x_2, \ldots, x_n$, and for $x_i = v$ there is no value $w \in D_j$, for $j > i$, such that $C_{ij}(v, w)$ is satisfied, then backtracking will try all values for x_j, fail and try all values for x_{j-1} (and for each value of x_{j-1} again all values for x_j), and so on until it tries all combinations of values for $x_{i+1}, \ldots, x_j$ before finally discovering that v is not a possible value for x_i.

   The identical failure process may be repeated for all other sets of values for $x_1, \ldots, x_{i-1}$ with $x_i = v$.

Local Consistency

- Consider CSP with unary and binary constraints only.
- Constraint graph G
  - For each variable x_i a node i.
  - For each pair of variables x_i, x_j occurring in the same binary constraint, two arcs (i, j) and (j, i).
- The node i is consistent if $C_i(v)$, for all $v \in D_i$.
- The arc (i, j) is consistent, if for all $v \in D_i$ with $C_i(v)$ there exists $w \in D_j$ with $C_j(w)$ such that $C_{ij}(v, w)$.
- The graph is node consistent resp. arc consistent if all its nodes (resp. arcs) are consistent.

Arc Consistency

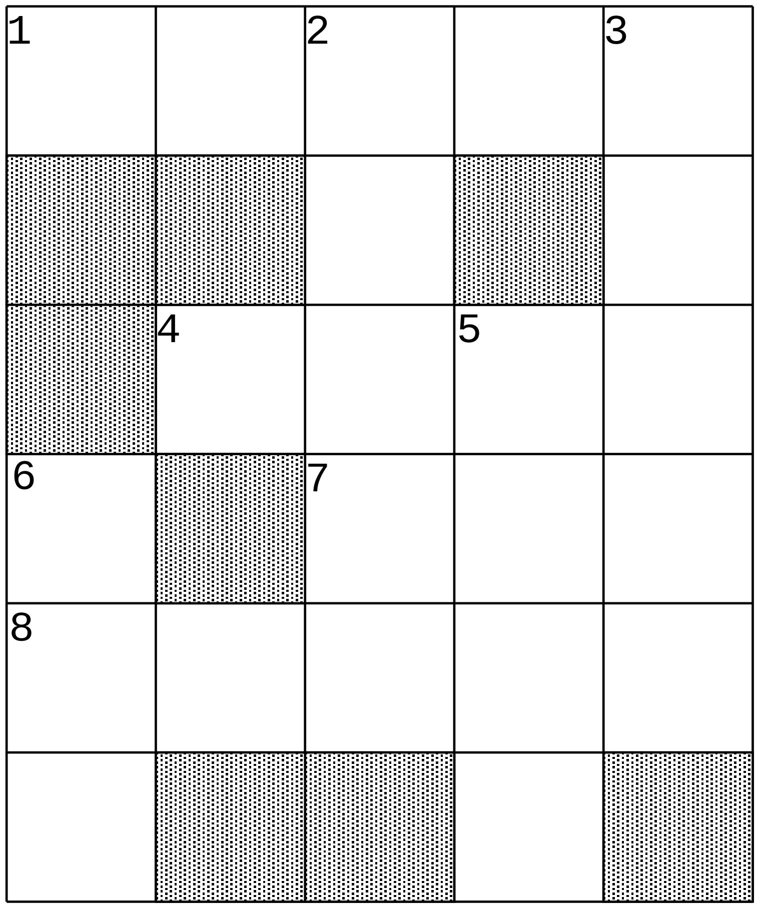
```
Algorithm AC-3 (Mackworth 77):
begin
   for i ← 1 until n do D_i ← {v ∈ D_i | C_i(v)};
   Q ← {(i,j) | (i,j) ∈ arcs(G), i ≠ j}
   while Q not empty do
      begin
         select and delete an arc (i,j) from Q;
         if REVISE(i,j) then
               Q ← Q ∪ {(k,i) | (k,i) ∈ arcs(G), k ≠ i, k ≠ j}
      end
end
```

Arc Consistency (2)

```
procedure REVISE(i,j):
begin
   DELETE ← false
   for each v ∈ D_i do
      if there is no w ∈ D_j such that C_ij(v,w) then
         begin
            delete v from D_i;
            DELETE ← true
         end;
   return DELETE
end
```

Crossword Puzzle

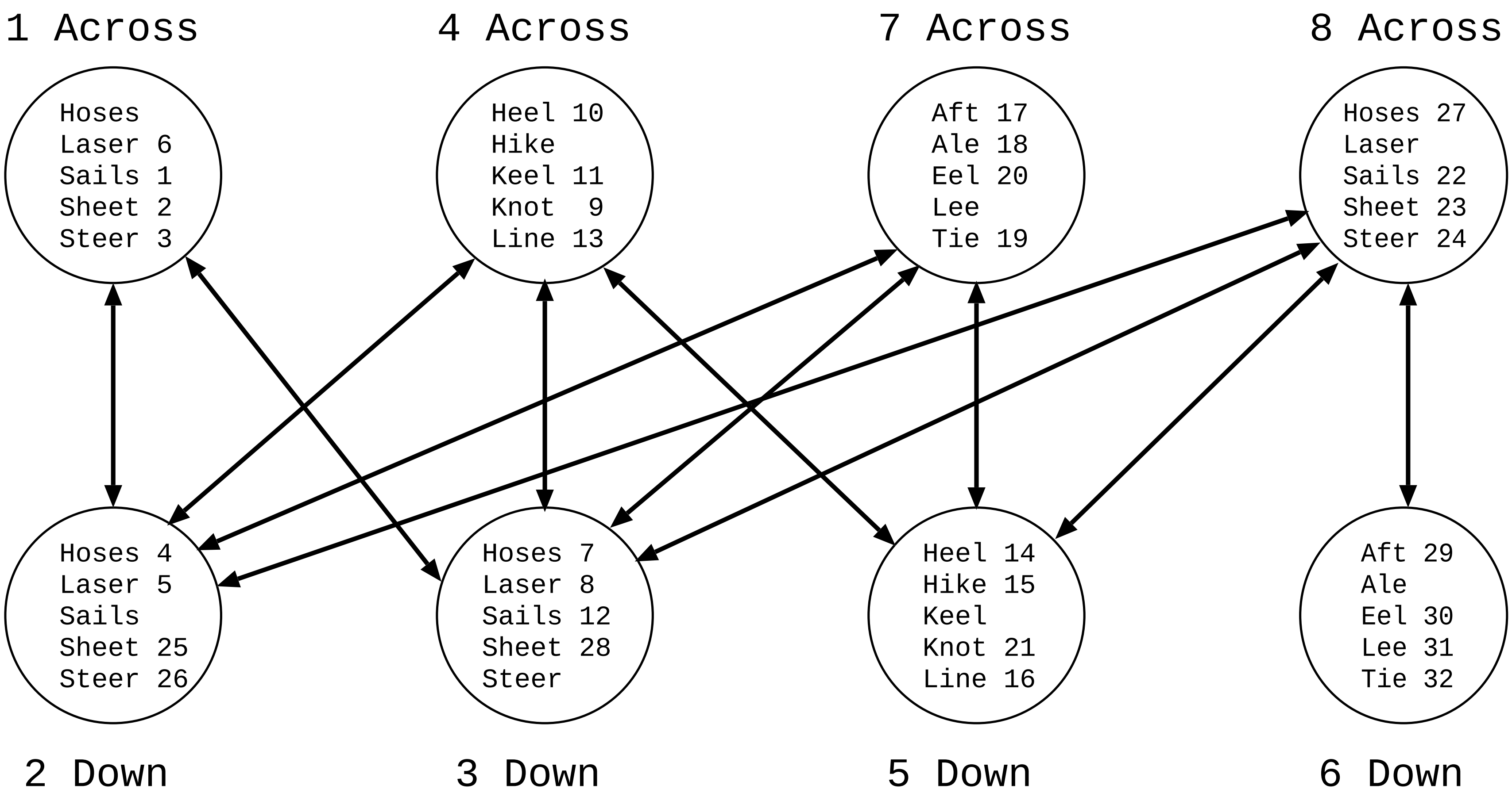
Dechter 92



Word List

Aft
Ale
Eel
Heel
Hike
Hoses
Keel
Knot
Laser
Lee
Line
Sails
Sheet
Steer
Tie

Solution


1 Across
Hoses
Laser 6
Sails 1
Sheet 2
Steer 3
4 Across
Heel 10
Hike
Keel 11
Knot 9
Line 13
7 Across
Aft 17
Ale 18
Eel 20
Lee
Tie 19
8 Across
Hoses 27
Laser
Sails 22
Sheet 23
Steer 24
Hoses 4
Laser 5
Sails
Sheet 25
Steer 26
Hoses 7
Laser 8
Sails 12
Sheet 28
Steer
Heel 14
Hike 15
Keel
Knot 21
Line 16
Aft 29
Ale
Eel 30
Lee 31
Tie 32
2 Down
3 Down
5 Down
6 Down

Lookahead

Apply local consistency dynamically during search

- Forward Checking: After assigning to x the value v, eliminate for all uninstantiated variables y the values from D_y that are incompatible with v.

- Partial Lookahead: Establish arc consistency for all (y, y'), where y, y' have not been instantiated yet and y will be instantiated before y'.

- Full Lookahead: Establish arc consistency for all uninstantiated variables.

n-Queens Problem

Place n queens in an $n \times n$ chessboard such that no two queens threaten each other.

- Variables $x_i, i = 1, \ldots, n$ with domain $D_i = \{1, \ldots, n\}$ indicating the column of the queen in line i.

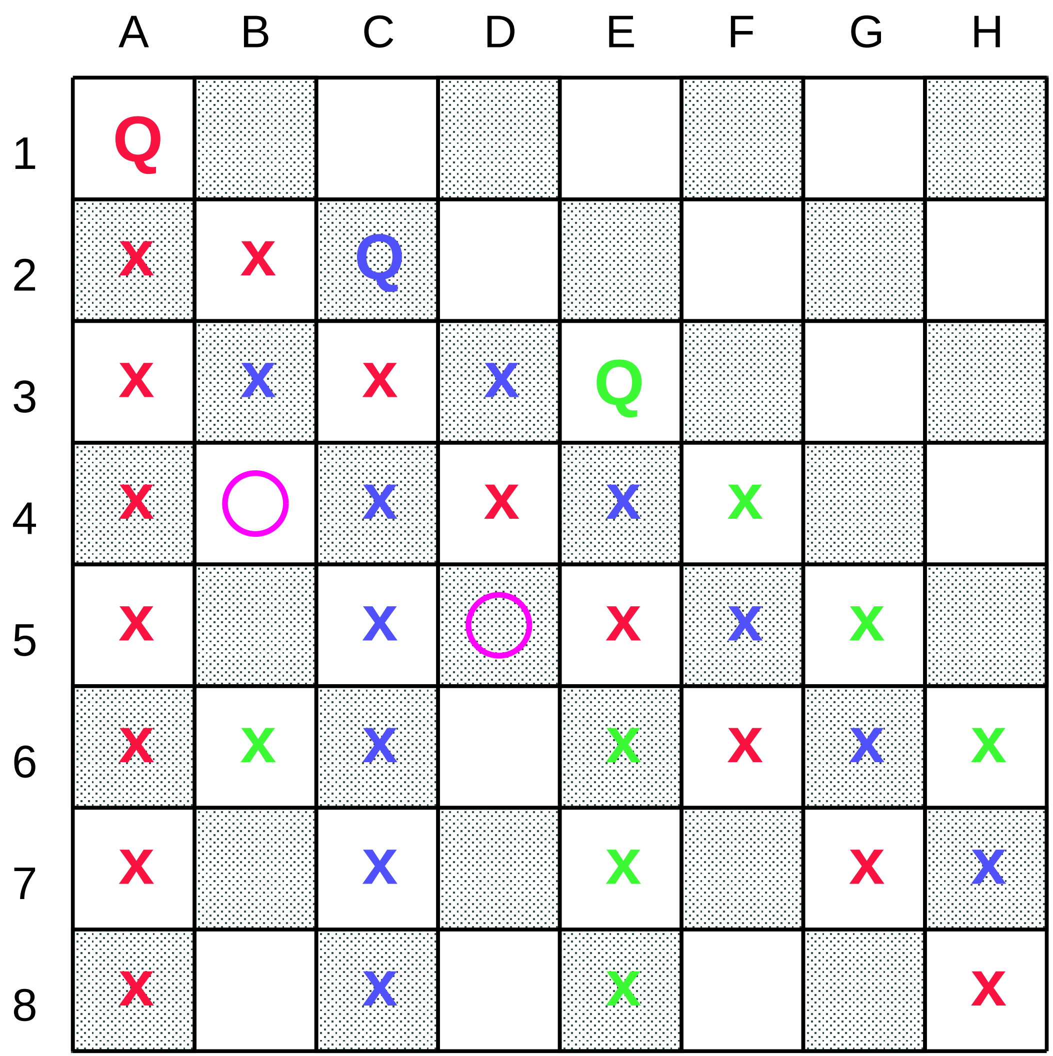
- Constraints
  - $x_i \neq x_j$, for $1 \leq i < j \leq n$ (vertical)
  - $x_i \neq x_j + (j - i)$, for $1 \leq i < j \leq n$ (diagonal 1)
  - $x_i \neq x_j - (j - i)$, for $1 \leq i < j \leq n$ (diagonal 2)

Forward Checking (2)

	A	B	C	D	E	F	G	H
1	Q							
2	X	X	Q					
3	X	X	X	X	Q			
4	X	Q	X	X	X	X		
5	X	X	X		X	X	X	
6	X	X	X	⊗	X	X	X	X
7	X	X	X		X		X	X
8	X	X	X		X	X		X

- 1A
  - 2C
    - 3E
      - 4BG
        - 5B
          - 6D
        - 5D
      - 4H
        - 5B
          - 6D
            - 7F
          - 6 (no more value)
        - 5D
      - 4 (no more value)
    - 3F

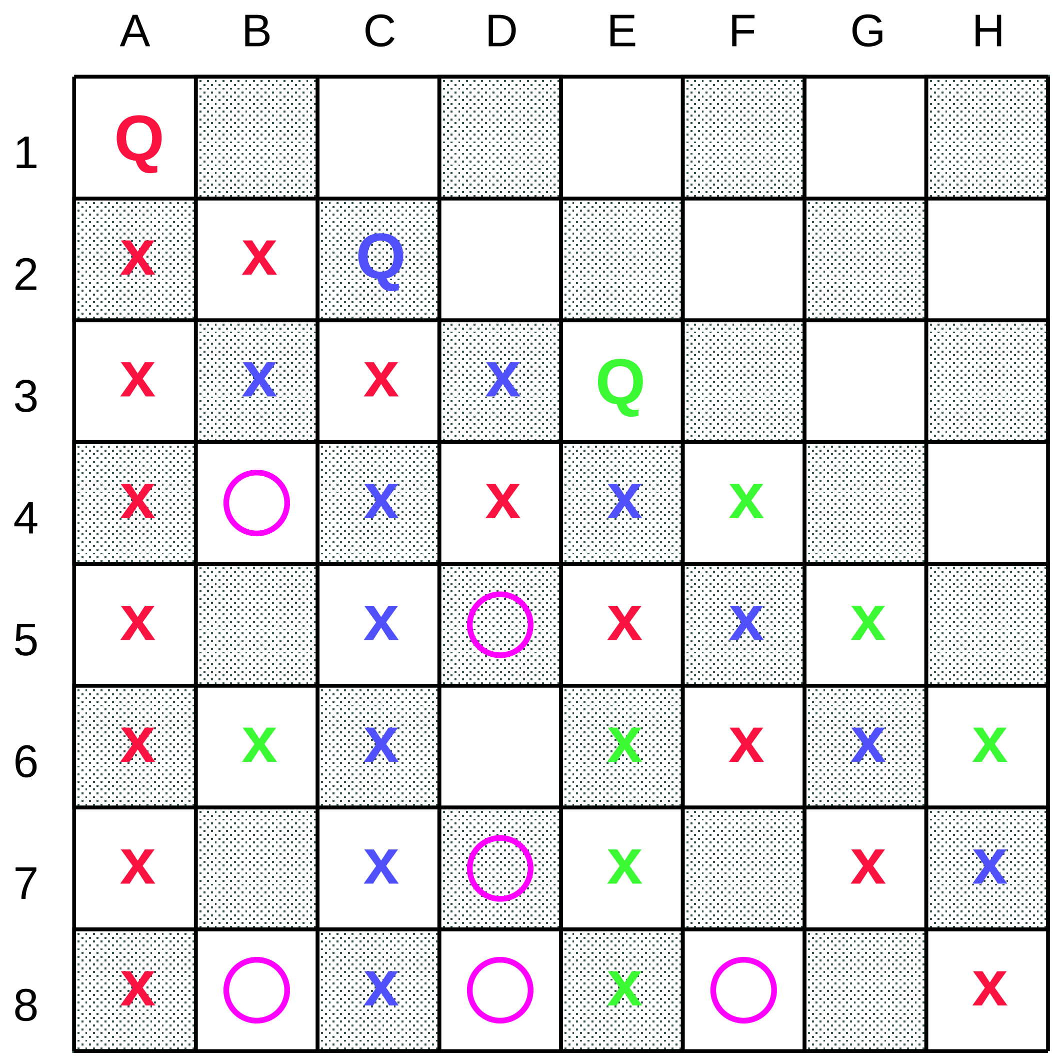
Partial Lookahead (3)



1A
  2C
    3E (delete 4B and 5D)
      4GH
        5B (no value left for 6)
    3F (delete 6D and 6E)
      4BH (failed, backtrack to 4)
    3G (delete 5D and 7E)
      4B

No value for queen 6

Full Lookahead (4)



1A
- 2C
  - 3E
  - 3F
  - 3G
  - 3H
- 2D
  - 3B
  - 3F

○ No value for queen 6

Typical structure of a constraint program

- Declare the variables and their domains
- State the constraints
- Enumeration (`labeling`)

The constraint solver achieves only local consistency.

In order to get global consistency, the domains have to be enumerated.

Labeling

- Assigning to the variables their possible values and constructing the corresponding search tree.

- Important questions

  1. In which order should the variables be instantiated (variable selection) ?

  2. In which order should the values be assigned to a selected variable (value selection) ?

- Static vs. dynamic orderings

- Heuristics

Dynamic variable/value orderings

- Variable orderings
  - ▷ Choose the variable with the smallest domain "first fail"
  - ▷ Choose the variable with the smallest domain that occurs in most of the constraints "most constrained"
  - ▷ Choose the variable which has the smallest/largest lower/upper bound on its domain.
- Value orderings
  - ▷ Try first the minimal value in the current domain.
  - ▷ Try first the maximal value in the current domain.
  - ▷ Try first some value in the middle of the current domain.

Constraint programming systems

System	Avail.	Constraints	Language	Web site
B-prolog	comm.	FinDom	Prolog	www.probp.com
CHIP	comm.	FinDom, Boolean, Linear $\mathbb{Q}$ Hybrid	Prolog, C, C++	www.cosytec.com
Choco	free	FinDom	Claire	choco-constraints.net
Eclipse	free non-profit	FinDom, Hybrid	Prolog	www.icparc.ic.ac.uk/ eclipse/
GNU Prolog	free	FinDom	Prolog	gnu-prolog.inria.fr
IF/Prolog	comm.	FinDom Boolean, Linear $\mathbb{R}$	Prolog	www.ifcomputer.co.jp
ILOG	comm.	FinDom, Hybrid	C++, Java	www.ilog.com
NCL	comm.	FinDom		www.enginest.com
Mozart	free	FinDom	Oz	www.mozart-oz.org
Prolog IV	comm.	FinDom, nonlinear intervals	Prolog	www.prologia.fr
Sicstus	comm.	FinDom, Boolean, linear $\mathbb{R}/\mathbb{Q}$	Prolog	www.sics.se/sicstus/

Integer vs. constraint programming

Practical Problem Solving

- Model building : Language

- Model solving : Algorithms

IP vs. CP : Language

	IP	CP
Variables	0-1	Finite domain
Constraints	Linear equations and inequalities	Arithmetic constraints Symbolic/global constraints

Example

- Variables : $x_1, \ldots, x_n \in \{0, \ldots, m-1\}$
- Constraint : Pairwise different values

Example (2)

- Integer programming: Only linear equations and inequalities

$$
\begin{aligned}
x_i \neq x_j \quad & \Longleftrightarrow \quad x_i < x_j \quad \vee \quad x_i > x_j \\
& \Longleftrightarrow \quad x_i \leq x_j - 1 \quad \vee \quad x_i \geq x_j + 1
\end{aligned}
$$

- Eliminating disjunction

$$
x_i - x_j + 1 \leq m y_i, \quad x_j - x_i + 1 \leq m y_j, \quad y_i + y_j = 1,
$$
$$
y_i, y_j \in \{0, 1\}, \quad 0 \leq x_i, x_j \leq m - 1,
$$

- New variables: $z_{ik} = 1$ iff $x_i = k$, $i = 1, \ldots, n$, $k = 0, \ldots, m - 1$

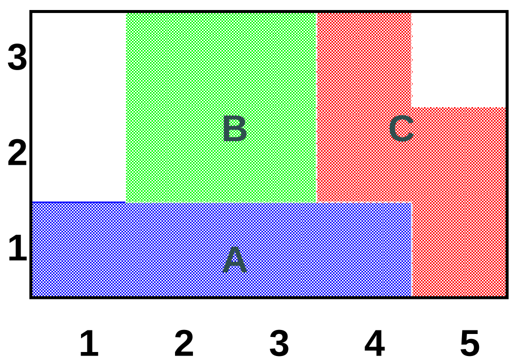
$$
z_{i0} + \cdots + z_{im-1} = 1, \quad z_{1k} + \cdots + z_{nk} \leq 1,
$$

- Constraint programming $\rightsquigarrow$ symbolic constraint

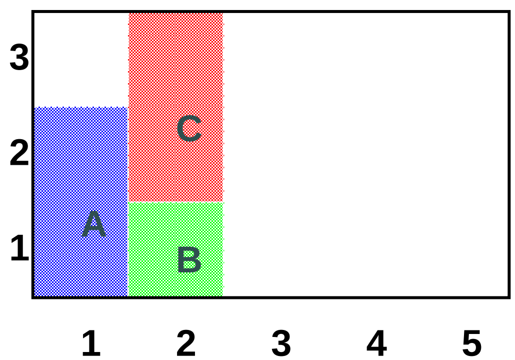
$$
\texttt{alldifferent}(x_1, \ldots, x_n)
$$

Symbolic/global constraints

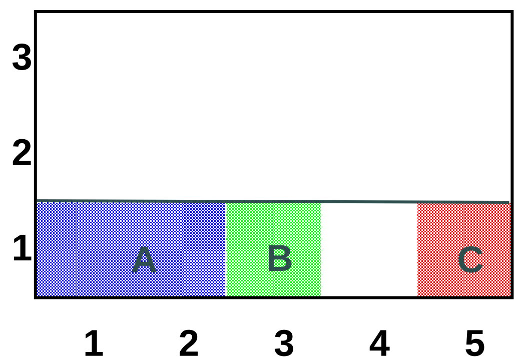
- `alldifferent`$([x_1, \ldots, x_n])$
- `cumulative`$([s_1, \ldots, s_n], [d_1, \ldots, d_n], [r_1, \ldots, r_n], c, e)$.
  - n tasks: starting time s_i, duration d_i, resource demand r_i
  - resource capacity c, completion time e.




```
cumulative([1,2,4],
           [4,2,2],
           [1,2,2], 3)
```




```
cumulative([1,2,2],
           [1,1,1],
           [2,1,2], 3)
```

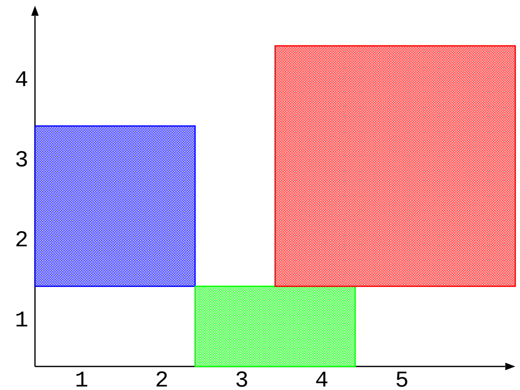



```
cumulative([1,3,5],
           [2,1,1],
           [1,1,1], 1)
```

Diffn Constraint

Beldiceanu/Contejean'94

- Nonoverlapping of n-dimensional rectangles $[O_1, \ldots, O_n, L_1, \ldots, L_n]$, where O_i resp. L_i denotes the origin resp. length in dimension i
- diffn$([[O_{11}, \ldots, O_{1n}, L_{11}, \ldots, L_{1n}], \ldots, [O_{m1}, \ldots, O_{mn}, L_{m1}, \ldots, L_{mn}]])$



`diffn([[1,2,2,2],[3,1,2,1],[4,2,3,3]])`

- General form: `diffn(Rectangles,Min_Vol,Max_Vol,End,Distances,Regions)`

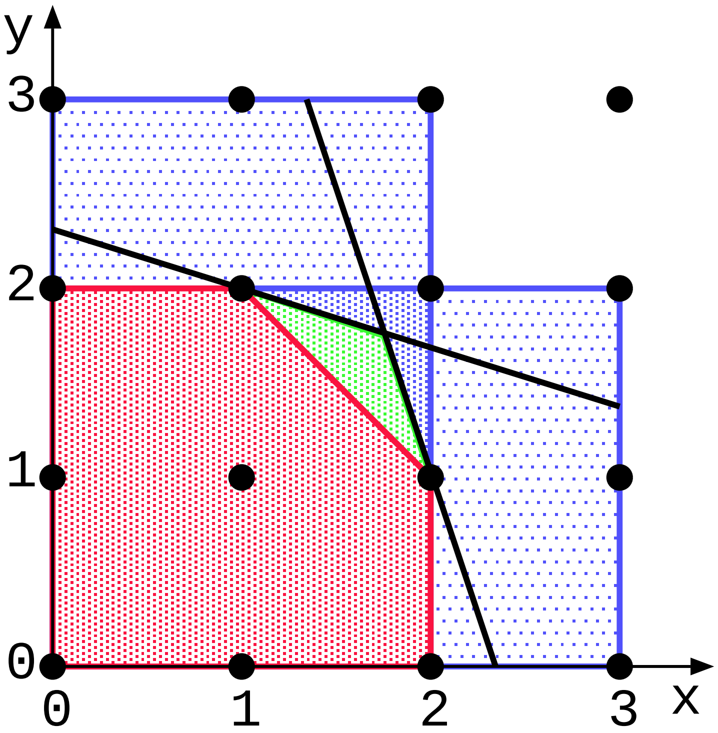
IP vs. CP : Algorithms

	IP	CP
Inference	Linear programming Cutting planes	Domain filtering Constraint propagation
Search	Branch-and-relax Branch-and-cut	Branch-and-bound
Bounds on the objective function	Two-sided	One-sided

Local vs. global reasoning

Linear arithmetic constraints

$$\begin{aligned} 3x + y &\leq 7, \\ 3y + x &\leq 7, \\ x + y &= z, \\ x, y &\in \{0, \dots, 3\} \end{aligned}$$



CP $x, y \leq 2, z \leq 4$

LP $x, y \leq 2, z \leq 3.5$

IP $x, y \leq 2, z \leq 3$

Global reasoning in CP ? $\rightsquigarrow$ global constraints !

Global reasoning in CP

Example

- $x_1, x_2, x_3 \in \{0, 1\}$
- pairwise different values
- Local consistency : 3 disequalities : $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$
  $\rightsquigarrow x_1, x_2, x_3 \in \{0, 1\}$, i.e., no domain reduction is possible
- Global constraint : `alldifferent`(x_1, x_2, x_3)
  $\rightsquigarrow$ detects infeasibility (uses bipartite matching)

Global reasoning in CP : inside global constraints

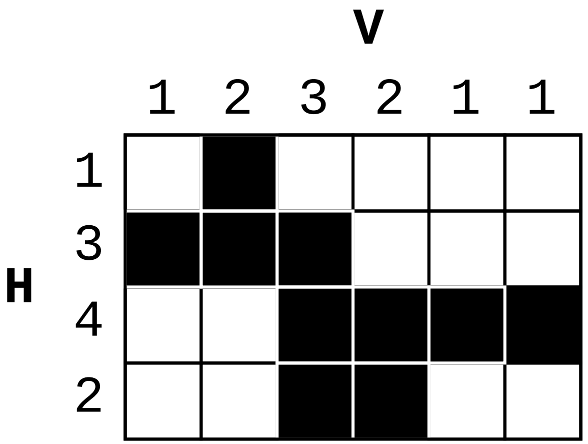
Summary

	ILP	CP(FD)
Language	Linear arithmetic —	Arithmetic constraints Symbolic constraints
Algorithms	Global consistency (LP) Cutting planes	Local consistency Domain reduction
	Branch-and-bound Branch-and-cut	User-defined enumeration

- Symbolic constraints $\rightsquigarrow$ more expressivity + more efficiency
- Unifying framework for CP and IP: Branch-and-infer (Bockmayr/Kasper 98)

Discrete Tomography

- Binary matrix with *m* rows and *n* columns
  - Horizontal projection numbers $(h_1, \ldots, h_m)$
  - Vertical projection numbers $(v_1, \ldots, v_n)$
- Properties
  - Horizontal convexity (h)
  - Vertical convexity (v)
  - Connectivity (polyomino) (p)



- Complexity (Woeginger'01)
  - polynomial: ( ), (p,v,h)
  - NP-complete: (p,v), (p,h), (v,h), (v), (h), (p)

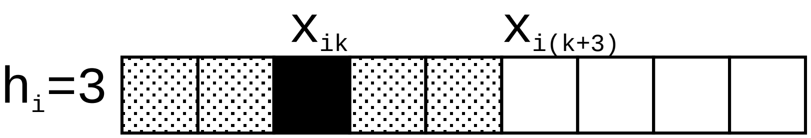
IP Model

- Variables $x_{ij} = \begin{cases} 0 & \text{cell(i,j) is labeled white} \\ 1 & \text{cell(i,j) is labeled black} \end{cases}$

- Constraints I: Projections

$$\sum_{j=1}^{n} x_{ij} = h_i, \qquad \sum_{i=1}^{m} x_{ij} = v_j$$

- Constraints II: Convexity

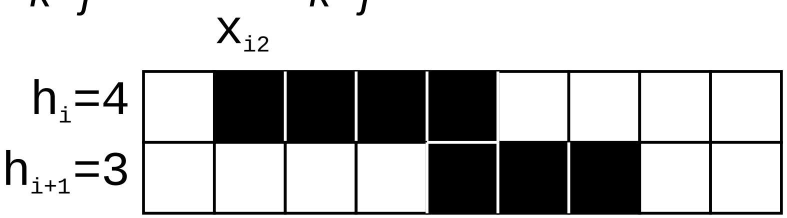


$$h_i x_{ik} + \sum_{l=k+h_i}^{n} x_{il} \leq h_i, \qquad v_j x_{kj} + \sum_{l=k+v_j}^{m} x_{lj} \leq v_j,$$

IP Model (contd)

- Constraints III: Connectivity

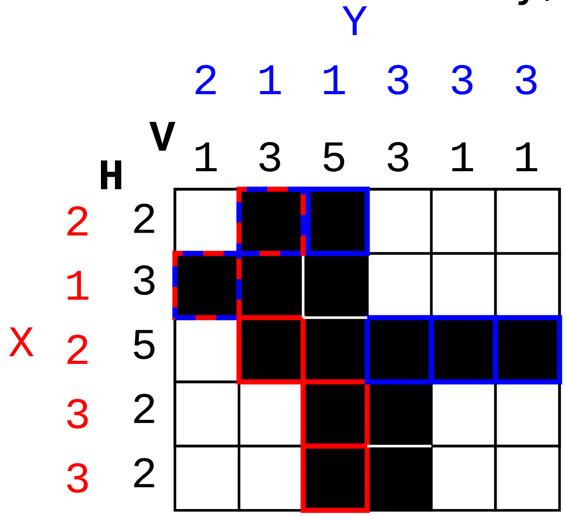
$$\sum_{k=j}^{j+h_i-1} x_{ik} - \sum_{k=j}^{j+h_i-1} x_{(i+1)k} \leq h_i - 1$$



- Various linear arithmetic models possible, e.g. convexity
- Enormous differences in size and running time, e.g. 1 day vs. $<$ 1 sec
- Large number of constraints ($\sim 3mn$ in the above model)

Finite Domain Model

- Variables
  - x_i start of horizontal convex block in row i, for $1 \leq i \leq m$
  - y_j start of vertical convex block in column j, for $1 \leq j \leq n$



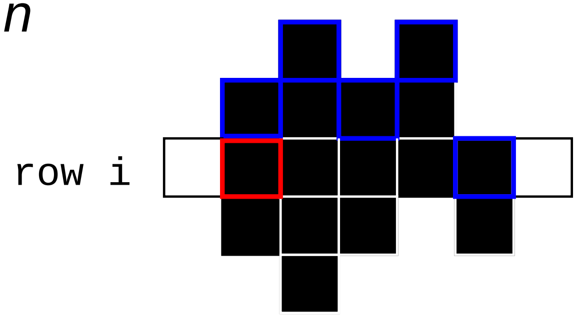
- Domain
  - $x_i \in [1, \ldots, n - h_i + 1]$, for $1 \leq i \leq m$
  - $y_j \in \left[1, \ldots, m - v_j + 1\right]$, for $1 \leq j \leq n$

Conditional Propagation

- Projection/Convexity modelled by FD variables
- Compatibility of x_i and y_j

$$x_i \le j < x_i + h_i \iff y_j \le i < y_j + v_j$$

for $1 \le i \le m$ and $1 \le j \le n$

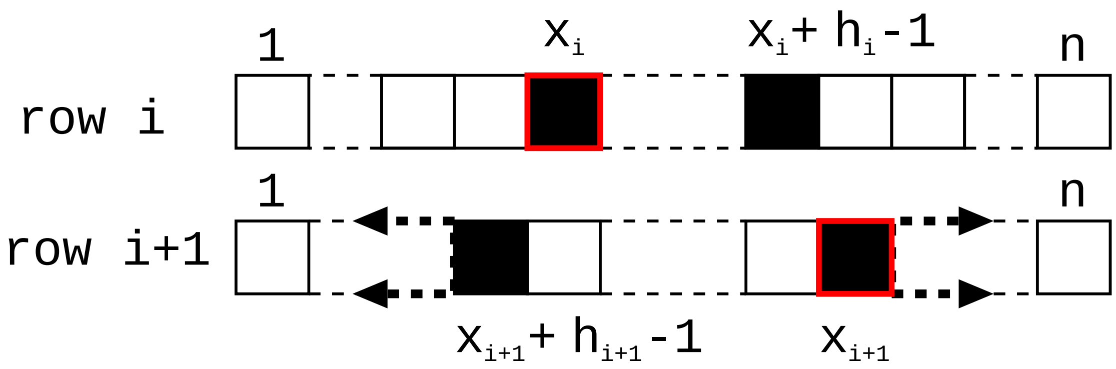


- Conditional propagation

$$\texttt{if } x_i \le j \texttt{ then } (\texttt{if } j < x_i + h_i \texttt{ then } (y_j \le i,\ i < y_j + v_j))$$

Finite Domain Model (contd)

- Connectivity



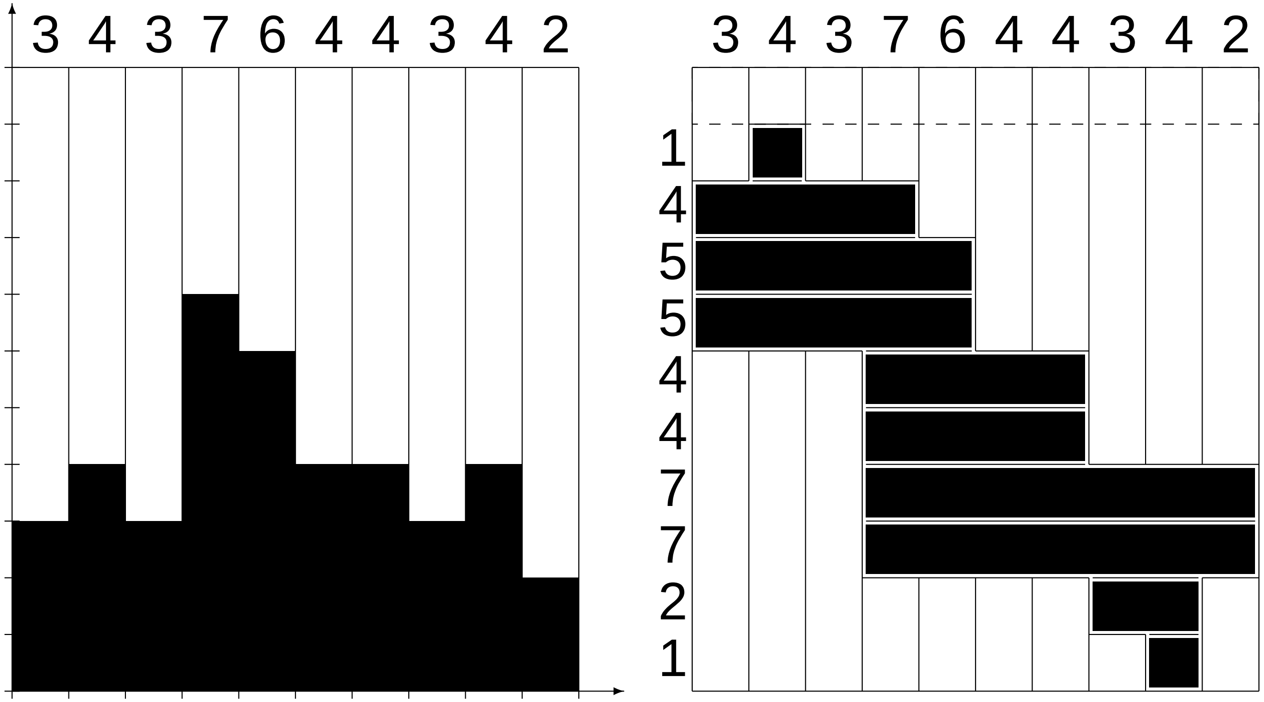
- Block i must start before the end of block $i + 1$

$$x_i \leq x_{i+1} + h_{i+1} - 1, \text{ for } 1 \leq i \leq m - 1$$

- Block $i + 1$ must start before the end of block i

$$x_{i+1} \leq x_i + h_i - 1, \text{ for } 1 \leq i \leq m - 1$$

Cumulative



2d and 3d Diffn Model