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Linear Algebra

We recall some basic definitions from linear algebra.

- A vector $x \in \mathbb{R}^n$ is a *linear combination* of the vectors $x^1, \ldots, x^p \in \mathbb{R}^n$ if

$$x = \sum_{i=1}^{p} \lambda_i x^i, \text{ for some } \lambda_1, \ldots, \lambda_p \in \mathbb{R}$$

- If in addition, $\lambda_1, \ldots, \lambda_p = 1$, x is an *affine* combination of $x^1, \ldots, x^p$.
- For $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, the *linear* (resp. *affine*) *hull* of X, denoted by $\mathrm{lin}(X)$ (resp. $\mathrm{aff}(X)$), is the set of all linear (resp. affine) combinations of finitely many vectors of X.
- A set $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$, is called *linearly independent* (resp, *affinely independent*) if no vector $x \in X$ is expressible as a linear (resp. affine) combination of the vectors in $X \setminus \{x\}$, otherwise X is called *linearly dependent* (resp. *affinely dependent*).
- The *rank* (resp *affine rank*) of X, denoted by $\mathrm{rank}(X)$ (resp $\mathrm{arank}(X)$), is the cardinality of the largest linearly (resp. affinely) independent subset of X.
- If $0 \in \mathrm{aff}(X)$ then $\mathrm{arank}(X) = \mathrm{rank}(X) + 1$, otherwise $\mathrm{arank}(X) = \mathrm{rank}(X)$.
- By definition, the *dimension* of a set $X \subseteq \mathbb{R}^n$ is the <u>maximum</u> number of <u>affinely independent</u> vectors in X minus one, i.e., $\dim(X) = \mathrm{arank}(X) - 1$.
- If $0 \in \mathrm{aff}(X)$ then $\dim(X) = \mathrm{rank}(X)$, otherwise $\dim(X) = \mathrm{rank}(X) - 1$.

Example:
Let

$$P = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1\},$$

$$F_1 = P \cap \{x \in \mathbb{R}^2 \mid x_1 = 0\},$$

$$F_2 = P \cap \{x \in \mathbb{R}^2 \mid x_1 = 1\}.$$

We have

$$\begin{aligned}
&\text{aff}(P) = \mathbb{R}^2, \\
&\text{aff}(F_1) = \{x \in \mathbb{R}^2 \mid x_1 = 0\}, \\
&\text{aff}(F_2) = \{x \in \mathbb{R}^2 \mid x_1 = 1\}.
\end{aligned}$$

- We can easily see that the vectors $(1,0)^T \in P$ and $(0,1)^T \in P$ are linearly independent in P. Therfeore, $\text{rank}(P) = 2$. Since $0 \in \text{aff}(P)$, we have $\dim(P) = \text{rank}(P) = 2$ (P is full-dimensional).
- F_1 and F_2 are two faces of P defined by the valid inequalities $0 \leq x_1$ and $x \leq 1$, respectively.
- No other vector in F_1 is linearly independent of the vector $(0,1)^T \in F_1$. Then, $\text{rank}(F_1) = 1$. Since $0 \in \text{aff}(F_1)$, we have $\dim(F_1) = \text{rank}(F_1) = 1$.
- We can easily see that the vectors $(1,0)^T \in F_2$ and $(1,1)^T \in F_2$ are linearly independent in F_2. Therefore, $\text{rank}(F_2) = 2$. Since $0 \notin \text{aff}(F_2)$, we have $\dim(F_2) = \text{rank}(F_2) - 1 = 1$.

Hint:
For a given set of vectors $x^1, \ldots, x^p \in \mathbb{R}^n$, they are linearly independent if

$$\lambda_1 x^1 + \lambda_2 x^2 + \ldots, +\lambda_p x^p = 0$$

only when $\lambda_1 = \lambda_2 = \ldots, = \lambda_p = 0$.