Markov Chains (MC)

1. Basics

GA Let $(\Omega, \mathcal{F}, P)$ be a probability space, $E = \{1, 2, \dots, l\}$, $l \in \mathbb{N}$, a finite set (or E a countable set), and $(X_n)_{n\in\mathbb{N}_0}$ a sequence of random variables $X_i : \Omega \to E$, $i \in \mathbb{N}_0$.

1.1 Def $(X_n)_{n\in\mathbb{N}_0}$ is called discrete-time stochastic process with state space E.

If $X_k = i$, $i \in E$, $k \in \mathbb{N}_0$, the process is said to be in state i at time k.

For $\omega \in \Omega$, the E-valued sequence $(X_0(\omega), X_1(\omega), X_2(\omega), \dots)$ is called realization (or trajectory or sample path) associated with ω.

1.2 Remarks

1. Recall the notion of distribution $\mu : E \to \mathbb{R}$, $i \mapsto P(X=i)$ for random variable $X : \Omega \to E$.

   Notation: $P(X=i) = P(\{X=i\}) = P(\{\omega \in \Omega \mid X(\omega) = i\})$

1.3 Def

A discrete-time stochastic process $(X_n)_{n\in\mathbb{N}_0}$ is called Markov chain if the Markov property

(1) $P(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_n = i_n \mid X_{n-1} = i_{n-1})$

holds f.a. $n \in \mathbb{N}$, $i_0, \dots, i_n \in E$ (assuming both sides of equ. (1) are defined).

If $p_{ij}(n) := P(X_n = j \mid X_{n-1} = i)$ does not depend on n, then the MC is called (time) homogeneous.

1.4 Remark

The conditional probability $P(A \mid B)$ is not defined, if $P(B) = 0$.

In the following consider statements containing cond. prob. only when defined, and only consider homogeneous MC.

1.5 Def

For a MC (X_n), the matrix $\mathbb{P} = (p_{ij})_{i,j\in E} \in [0,1]^{l\times l}$ with $p_{ij} = P(X_1 = j \mid X_0 = i)$ is called transition matrix of (X_n).

A function $\alpha : E \to [0,1]$ with $\sum_{i=1}^{l} \alpha(i) = 1$ and $\alpha(i) = P(X_0 = i)$ is called initial distribution of (X_n).

1.6 Remark

p_{ij} is the one-step transition probability from state i to state j.

It follows that $p_{ij} \geq 0$ f.a. $i,j \in E$ and $\sum_{j=1}^{l} p_{ij} = 1$, i.e. $\mathbb{P}$ is a stochastic matrix.

1.7 Theorem The following statements are equivalent:

(1) (X_n) is a MC,

(2) it ex. $\mathbb{P} = (p_{ij})_{i,j \in E} \in [0,1]^{l \times l}$ with $P(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = p_{i_{n-1} i_n}$

f.a. $n \in \mathbb{N}$, $i_0, \dots, i_n \in E$.

(3) it ex. $\mathbb{P} = (p_{ij})_{i,j \in E} \in [0,1]^{l \times l}$, $\alpha \in [0,1]^{l}$ with $P(X_0 = i_0, \dots, X_n = i_n) = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$

f.a. $n \in \mathbb{N}_0$, $i_0, \dots, i_n \in E$.

1.8 Remarks

A MC can be defined by giving a state space, a corresponding stochastic matrix and an initial distribution.
If no initial distribution is specified, the transition matrix describes a family of MCs.

1.9 Ex Describe repression of a gene:

$E = \{0, 1\}$, $X_n = 0$ - gene free at time n
$X_n = 1$ - gene repressed at time n

Assumptions: 1. gene free at time n, then gene repressed at time $n+1$ with prob. $p \geq 0$

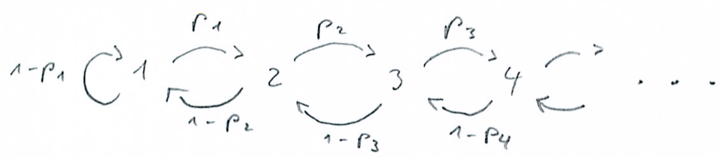
2. gene repressed at time n, then gene free at time $n+1$ with prob. $q \geq 0$

$$\rightarrow \quad \mathbb{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

1.10 Def Given the trans. matrix $\mathbb{P} \in [0,1]^{l \times l}$ of a MC (X_n) with state space E, the directed labeled graph G with vertex set E and edges (i, j, p_{ij}), $i, j \in E$, $p_{ij} \neq 0$ is called transition graph of (X_n).

1.11 Ex Random walk on $E = \mathbb{N}$

· choose $p_k \in (0,1)$ for all $k \in \mathbb{N}$.



2 Canonical representation

Goal: A representation of a MC amenable to simulation.

2.1 Theorem

Let $(Z_k)_{k\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d) random variables $Z_i : \Omega \to D$ (with $(D, \mathcal{D})$ some measurable space), and let $X_0 : \Omega \to E$ be a random variable independent of $Z_1, Z_2, \dots$

Consider a fct $f : E \times D \to E$.

Then the recurrence equation

$$(*) \quad X_{k+1} = f(X_k, Z_{k+1})$$

defines a homogeneous MC $(X_k)_{k\in\mathbb{N}_0}$ on state space E.

2.2 Remark

For the transition probabilities holds $p_{ij} = P(X_1 = j \mid X_0 = i) = P(f(i, Z_1) = j)$

2.3 Ex Random walk on $\mathbb{Z}$

$X_0 : \Omega \to \mathbb{Z}$ independent of i.i.d. random variables $Z_1, Z_2, \dots : \Omega \to \{-1, 1\}$

with $P(Z_k = 1) = q$, $P(Z_k = -1) = 1 - q$ for some $q \in (0,1)$.

MC (X_n) defined by $X_{n+1} = f(X_n, Z_{n+1}) = X_n + Z_{n+1}$, $f : \mathbb{Z} \times \{-1,1\} \to \mathbb{Z}$

2.4 Remark

1. Given (X_n), $\mathbb{P} = (p_{ij})_{i,j\in E}$ one can always define
   - (Z_k) sequence of i.i.d random variables uniformly distributed on $[0,1]$
   - $f : E \times [0,1] \to E$ with $f(i,z) = k$ if $\sum_{j=1}^{k-1} p_{ij} < z \leq \sum_{j=1}^{k} p_{ij}$

     (i.e. f not adopting value k if $\sum_{j=1}^{k-1} p_{ij} = \sum_{j=1}^{k} p_{ij}$, i.e. $p_{ik} = 0$).
2. Simulation of (X_n): obtain realization $(x_0, x_1, x_2, \dots)$
   - choose $x_0 \in E$ randomly (acc. to distribution of X_0)
   - choose $y_1, y_2, \dots \in D$ according to Z_1 (used representatively for all Z_k)
   - define $x_{k+1} = f(x_k, y_{k+1})$