Singular perturbation theory

Consider the system

$$\begin{aligned} \dot{x} &= f(x,y,\varepsilon), \qquad (*) \\ \varepsilon\dot{y} &= g(x,y,\varepsilon), \\ (x(0),y(0)) &= (x_0,y_0), \end{aligned}$$

with f,g sufficiently smooth, $0 \le \varepsilon \ll 1$.

- ▷ Slow variables x
- ▷ Fast variables y

⇝ separate time-scales (set $\varepsilon = 0$)

Hypotheses

(H1) there eyists a unique solution $y = h(x)$, sufficiently smooth, of $g(x,y,0) = 0$;
the matrix $\partial g/\partial y(y,h(x),0)$ has all eigenvalues with strictly negative real part;

(H2) the reduced system

$$\begin{aligned} \dot{x} &= f(x,h(x),0), \qquad (**) \\ x(0) &= x_0, \end{aligned}$$

has a solution $x^0(t)$ on an interval $[0,T]$, for some $T > 0$;

(H3) y_0 is in the basin of attraction of the steady state $h(x_0)$ of the fast system $\dot{\xi} = g(x_0,\xi,0)$.

Tikhonov's theorem

If these hypotheses are satisfied, system (*) admits a solution $(x^\varepsilon(t), y^\varepsilon(t))$ on $[0,T]$. In addition,

$$\lim_{\varepsilon\to 0+} x^\varepsilon(t) = x^0(t)$$

and

$$\lim_{\varepsilon\to 0+} y^\varepsilon(t) = y^0(t) = h(x^0(t)),$$

uniformly on any closed interval contained in $(0,T]$.

Slow manifold

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