# Model conditions and ASTG characterization

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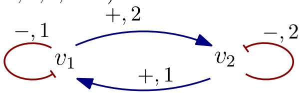
# IG, resource

## Definition

(Resource) The *resource* for a component $u \in V$ in a state $x \in X$ is defined as the set of predecessors of $u$ which in state $x$ have a present activation or an absent inhibition on $u$, *i.e.*,

$$\mathrm{Res}_u(x) = \left\{ v \in \mathrm{Pre}(u) \;\middle|\; \begin{array}{lll} \varepsilon(v,u) = + & \wedge & x_v \geq \vartheta(v,u) \\ \varepsilon(v,u) = - & \wedge & x_v < \vartheta(v,u) \end{array} \right\}.$$

$I = (V, E, \vartheta, \varepsilon, \max)$




$$\left.\begin{array}{l} Res_{v_1}(00) = \{v_1\} \\ Res_{v_1}(01) = \{v_1, v_2\} \\ Res_{v_1}(11) = \{v_2\} \\ Res_{v_1}(20) = \phi \end{array}\right] \subseteq \mathrm{Pre}(v_1) = \{v_1, v_2\}$$

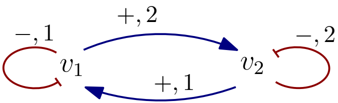
Figure: $V = \{v_1, v_2\}$, $E$: interactions, $\vartheta$: thresholds, $\varepsilon$: signs, and maximal activity levels, $max = [2, 2]$. Pre: predecessor.

# Logical parameter function

## Definition

(Logical parameter function) A *logical parameter function* $K$ is a function of the components and their resources. For all $u \in V$ and $\omega \subseteq \text{Pre}(u)$, the logical parameter $K(u,\omega) \in \{0,\dots,\max_u\}$ gives a value, to which a component $u$ in a state $x$ with resource $\text{Res}_u(x) = \omega$ tends.

$I$




$K$

| $\omega$ | $K(v_1,\omega)$ | $\omega$ | $K(v_2,\omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 1 | $\{v_2\}$ | 0 |
| $\{v_1\}$ | 1 | $\{v_1\}$ | 0 |
| $\{v_1,v_2\}$ | 2 | $\{v_1,v_2\}$ | 2 |

Figure: For $I$, $max = [2,2]$.

# State transition function and ASTG

## Definition

1. (State transition function) The *state transition function* $\delta : V \times X \to \{-1, 0, 1\}$ indicates how a component $u$ can change in a given state $x \in X$:

$$\delta(u, x) := \begin{cases} 1 & \text{if } x_u < K(u, \mathrm{Res}_u(x)), \\ 0 & \text{if } x_u = K(u, \mathrm{Res}_u(x)), \\ -1 & \text{if } x_u > K(u, \mathrm{Res}_u(x)). \end{cases}$$

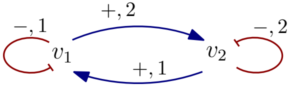
2. (Asynchronous state transition graph) The dynamics generated by a model $M$ is described by the *asynchronous state transition graph (ASTG)* $T_M = (X, S_M)$, where

$$S_M := \bigcup_{x \in X} \{(x, x + \delta(u, x)\mathbf{e}^u) \mid u \in V : \delta(u, x) \neq 0.\}$$

Here $\mathbf{e}^u$ is the $u$-th unit vector in $X$.

# Example: state transition function

$I$




$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 1 | $\{v_2\}$ | 0 |
| $\{v_1\}$ | 1 | $\{v_1\}$ | 0 |
| $\{v_1, v_2\}$ | 2 | $\{v_1, v_2\}$ | 2 |

For 3 states 00, 01, 02:

$\delta(v_1, 00) = 1$ $\quad \delta(v_2, 00) = 0$

$\delta(v_1, 01) = 1$ $\quad \delta(v_2, 01) = -1$

$\delta(v_1, 02) = 1$ $\quad \delta(v_2, 02) = -1$

For 3 states 00, 01, 02:

$x^0 = 00$

$Res_{v_1}(00) = \{v_1\}$ $\quad Res_{v_2}(00) = \{v_2\}$

$K(v_1, \{v_1\}) = 1$ $\quad K(v_1, \{v_1\}) = 0$

$\delta(v_1, 00) = 1$ $\quad \delta(v_2, 00) = 0$

$\vdots$

$Res_{v_1}(01) = \{v_1, v_2\}$ $\quad Res_{v_2}(01) = \{v_2\}$

$Res_{v_1}(02) = \{v_1, v_2\}$ $\quad Res_{v_1}(02) = \emptyset$

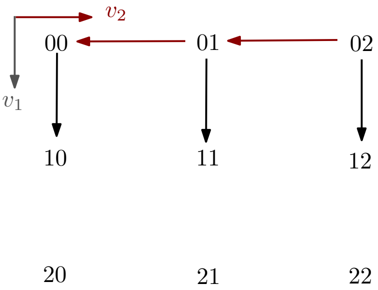
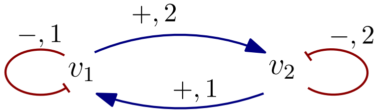



Figure: The state transition function $\delta(component, state)$

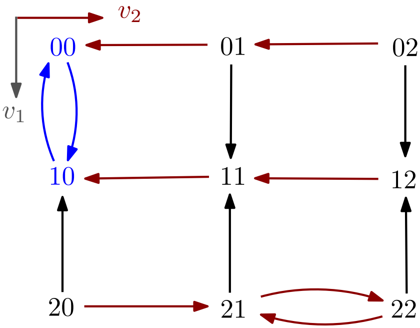
# ASTG

$I$




$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 1 | $\{v_2\}$ | 0 |
| $\{v_1\}$ | 1 | $\{v_1\}$ | 0 |
| $\{v_1, v_2\}$ | 2 | $\{v_1, v_2\}$ | 2 |

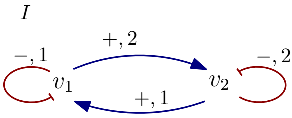



$\delta(v_1, 00) = 1$ $\delta(v_2, 00) = 0$

$\delta(v_1, 10) = -1$ $\delta(v_2, 10) = -1$

Figure: $M = (I, K)$ and the ASTG.

# $M$ and $T$: not a 1-to-1 correspondence.

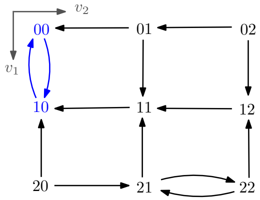



(a) $I$

$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 1 | $\{v_2\}$ | 0 |
| $\{v_1\}$ | 1 | $\{v_1\}$ | 0 |
| $\{v_1, v_2\}$ | 2 | $\{v_1, v_2\}$ | 2 |

(b) $K$




(c) $T$

$K'$

| $\omega$ | $K'(v_1, \omega)$ | $\omega$ | $K'(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 1 | $\{v_2\}$ | 0 |
| $\{v_1\}$ | 2 | $\{v_1\}$ | 1 |
| $\{v_1, v_2\}$ | 2 | $\{v_1, v_2\}$ | 2 |

(d) $K'$

Figure: (a) IG $I$. (b) Logical parameter function $K$. (c) Corresponding ASTG $T$. (d) Alternative logical parameter function $K'$. $M = (I, K)$ and $M' = (I, K')$, have the same ASTG.

# Isomorphic models [Lorenz2011]

## Definition

(Isomorphic models) Let $M_1 = (I, K_1)$ and $M_2 = (I, K_2)$ be two models with the same IG $I = (V, E, \varepsilon, \vartheta, max)$. $M_1$ and $M_2$ are *isomorphic* if

$$\text{ASTG}(M^1) \cong \text{ASTG}(M^2)$$

This isomorphism defines one kind of equivalence on the parameters of the models for an IG.

# Model condition: visibility condition

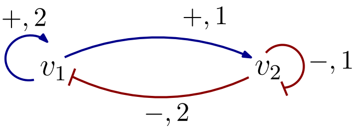
## Definition

In a model $M = (I, K)$, an interaction $(u, v) \in E$ is called *visible*, if there exists a resource $\omega \subseteq \mathrm{Pre}(v) \backslash \{u\}$ such that

$$K(v, \omega) \neq K(v, \omega \cup \{u\}).$$

A model $M = (I, K)$ satisfies the *visibility condition* if all edges $(u, v) \in E$ are visible.

$I$




$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 0 | $\{v_2\}$ | 1 |
| $\{v_1\}$ | 2 | $\{v_1\}$ | 2 |
| $\{v_1, v_2\}$ | 1 | $\{v_1, v_2\}$ | 2 |

Figure: Does $(v_2, v_1)$ satisfies the visibility condition? And the model $M$?

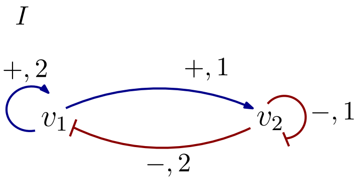
# Model condition: observability condition

## Definition

In a model $M = (I, K)$, an interaction $(u, v)$ is *observable* if there exists a resource $\omega \subseteq \mathrm{Pre}(v)\backslash\{u\}$ such that

$$K(v, \omega) < K(v, \omega \cup \{u\}),$$

A model $M = (I, K)$ satisfies the *observability condition* if all edges in $I$ are observable.




$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | $0$ | $\phi$ | $0$ |
| $\{v_2\}$ | $0$ | $\{v_2\}$ | $1$ |
| $\{v_1\}$ | $2$ | $\{v_1\}$ | $2$ |
| $\{v_1, v_2\}$ | $1$ | $\{v_1, v_2\}$ | $2$ |

Figure: Does $(v_2, v_1)$ satisfies the observability condition? And the model $M$?

# Model condition: Snoussi-condition [Snoussi1989]

## Definition

1. An *interaction* $(u, v) \in E$ of a model $M = (I, K)$ satisfies the *Snoussi-condition*, if

$$\forall \omega \subseteq \mathrm{Pre}(v) \backslash \{u\}, \;\; K(v, \omega) \leq K(v, \omega \cup \{u\}).$$

2. A *component* $v \in V$ of a model $M = (I, K)$ satisfies the *Snoussi-condition*, if

$$\forall \omega \subseteq \varsigma \subseteq \mathrm{Pre}(v), \;\; K(v, \omega) \leq K(v, \varsigma).$$

3. A *model* $M = (I, K)$ satisfies the *Snoussi-condition* if

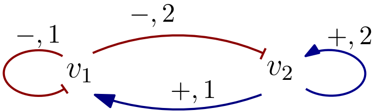
$$\forall v \in V, \forall \omega \subseteq \varsigma \subseteq \mathrm{Pre}(v), \;\; K(v, \omega) \leq K(v, \varsigma).$$

This means that when adding positive influences, no component tends to a lower value.

Snoussi1989, Qualitative dynamics of piecewise-linear differential equations: a discrete mapping approach, Dynamics and

# Example: Snoussi-condition

$I$




$K$

| $\omega$ | $K(v_1, \omega)$ | $\omega$ | $K(v_2, \omega)$ |
|---|---|---|---|
| $\phi$ | 0 | $\phi$ | 0 |
| $\{v_2\}$ | 2 | $\{v_2\}$ | 2 |
| $\{v_1\}$ | 1 | $\{v_1\}$ | 2 |
| $\{v_1, v_2\}$ | 1 | $\{v_1, v_2\}$ | 2 |

Figure: Does $M = (I, K)$ satisfies the Snoussi-condition?

A *model* $M = (I, K)$ satisfies the *Snoussi-condition* if $\forall v \in V$, $\forall \omega \subseteq \varsigma \subseteq \mathrm{Pre}(v)$, $K(v, \omega) \leq K(v, \varsigma)$.

# Programming Task: from model $\rightarrow$ ASTG

Think about the following tasks:

1. Data structures for the following:
   (a) IG $I = (V, E, \vartheta, \varepsilon, max)$;
   (b) Logical parameter function $K : K(v, \omega)$, for all $v \in V$ and for all $\omega \subseteq \mathrm{Pre}(v)$.
   (c) ASTG $T_M = (X, S_M)$, $X$: state space, $S_M$: transitions among all states in $X$.
2. Write a function, which can construct the ASTG from a given model.

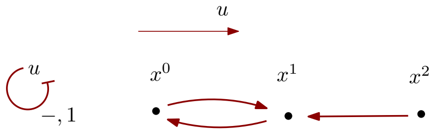
## Definition

($u$-row) Given $u \in V$, a *$u$-row* is a sequence of states $\tau^u = (x^0, \ldots, x^{max_u})$ in $X$, with $x^0_u = 0$ and $x^l = x^0 + l\mathbf{e}^u$, for all $l \in \{1, \ldots, max_u\}$, where $\mathbf{e}^u$ is the $u$-th unit vector.

## Lemma

*Given a $u$-row $\tau^u = (x^0, \ldots, x^{max_u})$ in $T_M$, $u$ under these states has at most two different resources.*

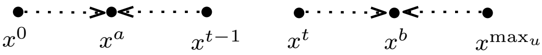
- If $(u, u) \notin E$, then $\mathrm{Res}_u(x^i) = \mathrm{Res}_u(x^0)$, for all $x^i \in \tau^u$.
- If $(u, u) \in E$ and $\vartheta(u, u) = t \in \{1, \ldots, max_u\}$, then: $\mathrm{Res}_u(x^i) = \mathrm{Res}_u(x^0)$ for all $i < t$, and $\mathrm{Res}_u(x^i) = \mathrm{Res}_u(x^{max_u})$ for all $i \geq t$.



# Proposition: $3$ row types [Lorenz2011, Lorenz2013]

Given an ASTG $T_M$, for each $u$-row $\tau^u = (x^0, \ldots, x^{max_u})$, exactly one of the following situations holds:

1. $\tau^u$ has the form:



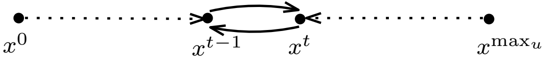

In the row structure above, there exists $a, b \in \{0, \ldots, max_u\}, a < b$ and $\delta(u, x^a) = \delta(u, x^b) = 0$. There exists $t \in \{1, \ldots, max_u\}$ so that the logical parameters $K_I(u, \mathrm{Res}_u(x^i)) = a$ for all $i < t$ and $K_I(u, \mathrm{Res}_u(x^i)) = b$ for all $i \geq t$. It follows that

$$
\begin{array}{ll}
(u, u) \in E & \vartheta(u, u) = t \\
K_I(u, \mathrm{Res}_u(x^0)) = a & K_I(u, \mathrm{Res}_u(x^{max_u})) = b
\end{array}
$$

$(u, u) \in E$. If the model $M = (I, K_I)$ satisfies the Snoussi-condition, then $\varepsilon(u, u) = +$.

Rename: *pos* type.

2. $\tau^u$ has the form:



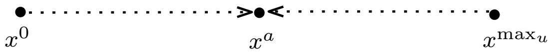

In the row structure above, there exists $t \in \{1, \ldots, max_u\}$, $\delta(u, x^i) = 1$ for all $i < t$ and $\delta(u, x^i) = -1$ for all $i \geq t$. Thus the logical parameters $K_I(u, \mathrm{Res}_u(x^i)) > t - 1$ for all $i \in \{0, \ldots, t-1\}$ and $K_I(u, \mathrm{Res}_u(x^i)) < t$ for all $i \in \{t, \ldots, max_u\}$. It follows that

$$\begin{array}{ll} (u, u) \in E & \vartheta(u, u) = t \\ K_I(u, \mathrm{Res}_u(x^0)) > t - 1 & K_I(u, \mathrm{Res}_u(x^{max_u})) < t \end{array}$$

$(u, u) \in E$. If the model $M = (I, K_I)$ satisfies the Snoussi-condition, then $\varepsilon(u, u) = -$.

Rename: *neg* type.

3. $\tau^u$ has the form:




In the row structure above, there exists $a \in \{0, \ldots, max_u\}$ that $K_I(u, \mathrm{Res}_u(x^i)) = a$ for all $i \in \{0, \ldots, max_u\}$. Let $\mathrm{Res}_u(x^0) = \omega$ and $\mathrm{Res}_u(x^{max_u}) = \varsigma$. There are three possibilities:

$$
\begin{array}{lll}
(1) & (u,u) \notin E, & K_I(u, \mathrm{Res}_u(x^0)) = a. \\
(2) & (u,u) \in E, & \vartheta(u,u) \geq a, \\
 & K_I(u, \mathrm{Res}_u(x^0)) = a, & K_I(u, \mathrm{Res}_u(x^{max_u})) < \vartheta(u,u). \\
(3) & (u,u) \in E, & \vartheta(u,u) < a, \\
 & K_I(u, \mathrm{Res}_u(x^0)) \geq \vartheta(u,u), & K_I(u, \mathrm{Res}_u(x^{max_u})) = a.
\end{array}
$$

A $u$-row in $T_M$ of *pos* or *neg* type also gives the threshold value $\vartheta(u,u)$. If there are no *pos* or *neg* type of $u$-rows in $T_M$, then $(u,u) \notin E$.

Rename: *open* type.

# Examples.

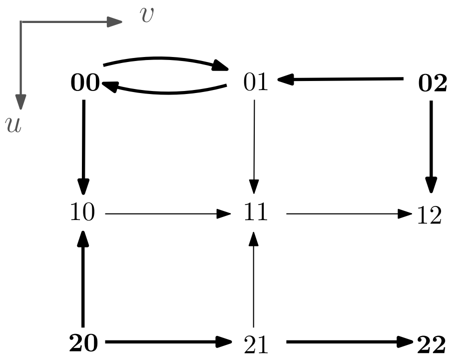


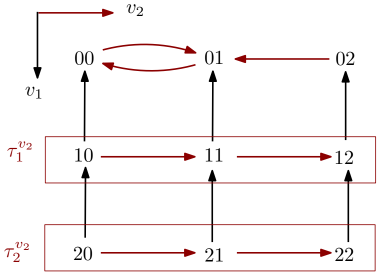

Figure: Examples, rows of *pos*, *neg* and *open* type.

- *neg* type: $(00, 01, 02)$.
- *pos* type: $(02, 12, 22)$.
- *open* type: $(00, 10, 20)$.

# Isomorphic $u$-rows

## Definition

Two $u$-rows $\tau_x^u = (x^0, \ldots, x^{max_u})$ and $\tau_y^u = (y^0, \ldots, y^{max_u})$ are *isomorphic* to each other, if for all $i \in \{0, \ldots, max_u\}$, for all $x^i \in \tau_x^u$, for all $y^i \in \tau_y^u$, there is $\mathrm{Res}_u(x^i) = \mathrm{Res}_u(y^i)$, which leads to $\delta(u, x^i) = \delta(u, y^i)$.



# Isomorphic groups of $u$-rows [Lorenz2011]

## Lemma

*Let* $u \neq v \in V$ *and* $x \in X$ *with* $x_v = x_u = 0$. *For every* $j \in \{0, \dots, max_v\}$, *let* $\tau_j^u$ *be the* $u$*-row with starting state* $x_j^0 = x^0 + j\mathbf{e}_v$.

1. *If* $(v, u) \notin E$, *all* $u$*-rows* $\tau_0^u, \dots, \tau_{max_v}^u$ *are isomorphic to each other.*
2. *If* $(v, u) \in E$, *then there are two groups of isomorphic* $u$*-rows. More precisely, the* $u$*-rows* $\tau_0^u, \dots, \tau_{\vartheta(v,u)-1}^u$ *are isomorphic to each other and the* $u$*-rows* $\tau_{\vartheta(v,u)}^u, \dots, \tau_{max_v}^u$ *are isomorphic to each other.*