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Optimization

WS 2016/17

Exercise 5

1. **Lagrangean Relaxation I**
   Consider the following problem

$$
\begin{array}{rrcrcr}
\min & 2x_1 & - & 3x_2 & & \\
\text{w.r.t.} & 3x_1 & - & 4x_2 & \leq & -6 \\
 & -x_1 & + & x_2 & \leq & 2 \\
 & 6x_1 & + & 2x_2 & \geq & 3 \\
 & 6x_1 & + & x_2 & \leq & 15 \\
 & & & x_1, x_2 & \geq & 0 \\
 & & & x_1, x_2 & \in & \mathbb{Z}
\end{array}
$$

   (a) Draw the corresponding polytope and determine graphically the optimal solution Z_{IP} of the original problem and Z_{LP}, the solution of the LP-relaxation.

   (b) Now apply lagrangean relaxation by relaxing the first inequality. Draw the polytope of the relaxed ILP. Determine the set X of feasible solutions for the relaxed problem.

   (c) The new objective function is then:

$$Z(P) = \min_{(x_1,x_2)\in X} 2x_1 - 3x_2 + p(6 + 3x_1 - 4x_2)$$

   Calculate $Z_D = \max_{p\geq 0} Z(p)$ and compare this value to Z_{IP} and Z_{LP}. (To obtain Z_D, draw the graphs of the function $f(p) = 2x_1 - 3x_2 + p(-6 - 3x_1 + 4x_2)$ for all $(x_1, x_2) \in X$. Do this by hand.)

   (d) repeat a-c for the objective functions $-x_1 + x_2$ and $-x_1 - x_2$ and compare Z_{LP}, Z_D, and Z_{IP}. To draw the graphs of the functions $f(p)$ use MATLAB.

2. **Lagrangean Relaxation II**
   Prove Lemma 1 (see script page 4001) stating that (in case of a minimization problem) if $\lambda \geq 0$, then $Z(\lambda) \leq Z_{IP}$, where Z_{IP} is the optimal value of an original ILP and $Z(\lambda)$ is the optimal value of the relaxed problem for a given value of the Lagrangean multiplier λ.

3. **MILP**

Given a metabolic network: a stoichiometric matrix $S \in \mathbb{R}^{m \times n}$, the set of indices corresponding to the irreversible reactions Irr, and lower and upper bounds for the fluxes of the reactions, lb and ub respectively.

Build a Mixed Integer Linear Program (MILP) such that:

- the system is in steady state
- the fluxes respect the lower and upper bounds
- the number of active reactions (reactions which carry flux) is minimal
- the trivial flux vector ($v = 0$) is not a feasible solution

Thus we search for a flux vector v s.t. the number of entries with $v_i \neq 0$ is minimal.

(a) suppose that all reactions are irreversible.

(b) there exists reversible reactions.

(*Hint:* To minimise over the number of active reactions, binary variables are needed. For each v_i there has to exist a binary a_i which indicates that v_i carries flux or not. Thus: $a_i = 0 \Leftrightarrow v_i = 0$. Try to formulate "$\Leftrightarrow$" with the help of linear constraints.)