Basic solutions

- $Ax \leq b$, $A \in \mathbb{R}^{m \times n}$, $rank(A) = n$.
- $M = \{1, \dots, m\}$ row indices, $N = \{1, \dots, n\}$ column indices
- For $I \subseteq M, J \subseteq N$ let A_{IJ} denote the submatrix of A defined by the rows in I and the columns in J.
- $I \subseteq M, |I| = n$ is called a *basis of A* iff $A_{I*} = A_{IN}$ is non-singular.
- In this case, $A_{I*}^{-1} b_I$, where b_I is the subvector of b defined by the indices in I, is called a *basic solution.*
- If $x = A_{I*}^{-1} b_I$ satisfies $Ax \leq b$, then x called a *basic feasible solution* and I is called a *feasible basis.*

Algebraic characterization of vertices

Theorem

Given the non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where rank$(A) = n$, a vector $v \in \mathbb{R}^n$ is a vertex of P if and only if it is a basic feasible solution of $Ax \leq b$, for some basis I of A.

For any $c \in \mathbb{R}^n$, either the maximum value of $z = c^T x$ for $x \in P$ is attained at a vertex of P or z is unbounded on P.

Corollary

P has at least one and at most finitely many vertices.

Remark

In general, a vertex may be defined by several bases.

Simplex Algorithm: Algebraic version

- Suppose rank$(A) = n$ (otherwise apply Gaussian elimination).
- Suppose I is a feasible basis with corresponding vertex $v = A_{I*}^{-1} b_I$.
- Compute $u^T \stackrel{\text{def}}{=} c^T A_{I*}^{-1}$ (vector of n components indexed by I).
- If $u \geq 0$, then v is an optimal solution, because for each feasible solution x

$$c^T x = u^T A_{I*} x \leq u^T b_I = u^T A_{I*} v = c^T v.$$

- If $u \not\geq 0$, choose $i \in I$ such that $u_i < 0$ and define the direction $d \stackrel{\text{def}}{=} -A_{I*}^{-1} e_i$, where e_i is the i-th unit basis vector in $\mathbb{R}^I$.
- Next increase the objective function value by going from v in direction d, while maintaining feasibility.

Simplex Algorithm: Algebraic version (2)

1. If $Ad \not\leq 0$, the largest $\lambda \geq 0$ for which $v + \lambda d$ is still feasible is

$$\lambda^* = \min\{\frac{b_l - A_{l*} v}{A_{l*} d} \mid l \in \{1, \dots, m\}, A_{l*} d > 0\}. \quad \text{(PIV)}$$

   Let this minimum be attained at index k. Then $k \notin I$ because $A_{I*} d = -e_i \leq 0$.

   Define $I' = (I \setminus \{i\}) \cup \{k\}$, which corresponds to the vertex $v + \lambda^* d$.

   Replace I by I' and repeat the iteration.

2. If $Ad \leq 0$, then $v + \lambda d$ is feasible, for all $\lambda \geq 0$. Moreover,

$$c^T d = -c^T A_{I*}^{-1} e_i = -u^T e_i = -u_i > 0.$$

   Thus the objective function can be increased along d to infinity and the problem is unbounded.

Termination and complexity

- The method terminates if the indices i and k are chosen in the right way (such choices are called *pivoting rules*).
- Following the rule of Bland, one can choose the smallest i such that $u_i < 0$ and the smallest k attaining the minimum in (PIV).
- For most known pivoting rules, sequences of examples have been constructed such that the number of iterations is exponential in $m + n$ (e.g. Klee-Minty cubes).
- Although no pivoting rule is known to yield a polynomial time algorithm, the Simplex method turns out to work very well in practice.

Simplex : Phase I

- In order to find an *initial feasible basis*, consider the auxiliary linear program

$$\max\{y \mid Ax - by \leq 0, \ -y \leq 0, \ y \leq 1\}, \qquad \text{(Aux)}$$

  where y is a new variable.

- Given an arbitrary basis K of A, obtain a feasible basis I for (Aux) by choosing $I = K \cup \{m+1\}$. The corresponding basic feasible solution is 0.
- Apply the Simplex method to (Aux). If the optimum value is 0, then (LP) is infeasible. Otherwise, the optimum value has to be 1.
- If I' is the final feasible basis of (Aux), then $K' = I' \setminus \{m+2\}$ can be used as an initial feasible basis for (LP).

Application: Metabolic networks