Constraint Programming

Constraint Programming

- *Basic idea:* Programming with constraints, i.e. constraint solving embedded in a programming language
- *Constraints:* linear, non-linear, finite domain, Boolean, ...
- *Programming:* logic, functional, object-oriented, concurrent, imperative/declarative, ...
- Mathematical programming vs. computer programming
- *Systems:* Prolog III/IV, CHIP, ECLIPSE, ILOG, CHOCO, Gecode, JaCoP, MiniZinc, ...

Recommended reading: Lustig/Puget'01

Finite Domain Constraints

Constraint satisfaction problem (CSP)

- n variables $x_1, \dots, x_n$
- For each variable x_j a *finite domain* D_j of possible values, often $D_j \subset \mathbb{N}$.
- m constraints $C_1, \dots, C_m$, where $C_i \subseteq D_{i_1} \times \dots \times D_{i_{k_i}}$ is a relation between k_i variables $x_{i_1}, \dots, x_{i_{k_i}}$. Write also $C_{i_1,\dots,i_{k_i}}$.
- A *solution* is an assignment of a value $v_j \in D_j$ to x_j, for each $j = 1, \dots, n$, such that all relations C_i are satisfied.

Coloring Problem

- Decide whether a map can be colored by 3 colors such that neighboring regions get different colors.
- For each region a variable $\mathbf{x_j}$ with domain $D_j = \{\text{red, green, blue}\}$.
- For each pair of variables x_i, x_j corresponding to two neighboring regions, a constraint $\mathbf{x_i} \neq \mathbf{x_j}$.
- NP-complete problem.

Resolution by Backtracking

- Instantiate the variables in some order.
- As soon as all variables in a constraint are instantiated, determine its truth value.
- If the constraint is not satisfied, backtrack to the last variable whose domain contains unassigned values, otherwise continue instantiation.

Efficiency Problems

Mackworth 77

1. If the domain D_j of a variable x_j contains a value v that does not satisfy C_j, this will be the cause of repeated instantiation followed by immediate failure.
2. If we instantiate the variables in the order $x_1, x_2, \dots, x_n$, and for $x_i = v$ there is no value $w \in D_j$, for $j > i$, such that $C_{ij}(v, w)$ is satisfied, then backtracking will try all values for x_j, fail and try all values for x_{j-1} (and for each value of x_{j-1} again all values for x_j), and so on until it tries all combinations of values for $x_{i+1}, \dots, x_j$ before finally discovering that v is not a possible value for x_i.

   The identical failure process may be repeated for all other sets of values for $x_1, \dots, x_{i-1}$ with $x_i = v$.

Local Consistency

- Consider CSP with unary and binary constraints only.
- *Constraint graph G*
  - For each variable x_i a node i.
  - For each pair of variables x_i, x_j occurring in the same binary constraint, two arcs (i, j) and (j, i).
- The node i is *consistent* if $C_i(v)$, for all $v \in D_i$.
- The arc (i, j) is *consistent*, if for all $v \in D_i$ with $C_i(v)$ there exists $w \in D_j$ with $C_j(w)$ such that $C_{ij}(v, w)$.
- The graph is *node consistent* resp. *arc consistent* if all its nodes (resp. arcs) are consistent.

Arc Consistency

Algorithm AC-3 (Mackworth 77):

```
begin
   for i ← 1 until n do D_i ← {v ∈ D_i | C_i(v)};
   Q ← {(i,j) | (i,j) ∈ arcs(G), i ≠ j}
   while Q not empty do
      begin
         select and delete an arc (i,j) from Q;
         if REVISE(i,j) then
               Q ← Q ∪ {(k,i) | (k,i) ∈ arcs(G), k ≠ i, k ≠ j}
      end
end
```

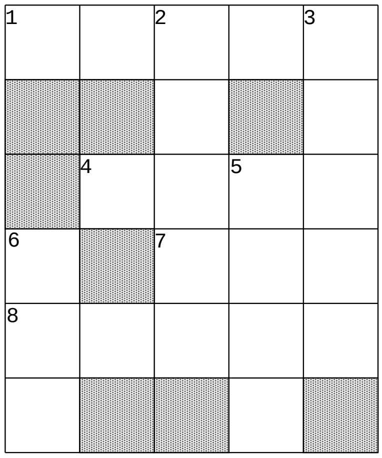
Arc Consistency (2)

```
procedure REVISE(i,j):
begin
   DELETE ← false
   for each v ∈ D_i do
      if there is no w ∈ D_j such that C_ij(v,w) then
         begin
            delete v from D_i;
            DELETE ← true
         end;
   return DELETE
end
```

Complexity: $O(d^3 e)$, with d an upper bound on the domain size and e the number of binary constraints, can be improved to $O(d^2 e)$.

Crossword Puzzle

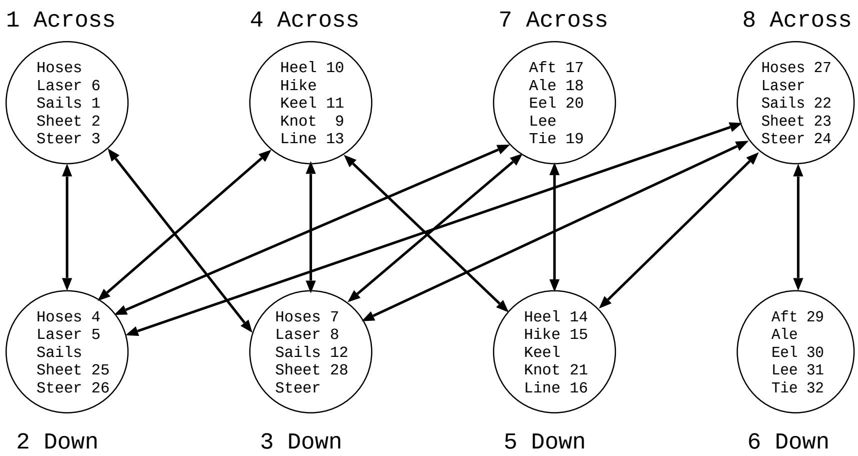
Dechter 92



Word List

Aft	Laser
Ale	Lee
Eel	Line
Heel	Sails
Hike	Sheet
Hoses	Steer
Keel	Tie
Knot	

Solution



Lookahead

Apply local consistency dynamically during search

- *Forward Checking:* After assigning to x the value v, eliminate for all uninstantiated variables y the values from D_y that are incompatible with v.
- *Partial Lookahead:* Establish arc consistency for all (y, y'), where y, y' have not been instantiated yet and y will be instantiated before y'.
- *Full Lookahead:* Establish arc consistency for all uninstantiated variables.