Reminder: Eigenvalues and eigenvectors

- Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $A \in \mathbb{F}^{n \times n}$ is a matrix.
- $\lambda \in \mathbb{F}$ is called an *eigenvalue* of A if there exists an *eigenvector* $v \in \mathbb{F}^n$, $v \neq 0$ such that $Av = \lambda v$.
- *Characteristic polynomial*

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

$\rightsquigarrow$ polynomial of degree n

- **Lemma.** λ is an eigenvalue of A if and only if $\chi_A(\lambda) = 0$.

Stability analysis of two-dimensional linear systems

- $\dot{y} = Ay$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ non-singular
- Characteristic equation:

$$\det(A - \lambda I) = \lambda^2 - (\underbrace{a+d}_{S})\lambda + (\underbrace{ad - bc}_{P}) = 0$$

- Eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(S \pm \sqrt{S^2 - 4P})$$

- Different cases depending on whether or not
  - λ_1, λ_2 are real or complex,
  - λ_1, λ_2 (resp. their real part) are positive or negative.

Possible cases

- $S^2 - 4P \geq 0$: Real roots λ_1, λ_2
  - $\lambda_1 \cdot \lambda_2 > 0$: Node
  - $\lambda_1 \cdot \lambda_2 < 0$: Saddle point
- $S^2 - 4P < 0$: Complex conjugate roots $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$
  - $\alpha \neq 0$: Focus
  - $\alpha = 0$: Center

Real eigenvalues $\lambda_1 \neq \lambda_2$

- Diagonalization $\dot{z} = Dz$, with $D = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- Solutions: $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$
- In phase plane: $|z_2| = c|z_1|^{\lambda_2/\lambda_1}$

- Case $\lambda_2/\lambda_1 > 0$: parabolic orbits *Node*
  - $\lambda_1, \lambda_2 < 0$: *stable*
  - $\lambda_1, \lambda_2 > 0$: *unstable*
- Case $\lambda_2/\lambda_1 < 0$: hyperbolic orbits *Saddle point*

Real eigenvalues $\lambda_1 = \lambda_2$

- $\lambda_1 = \lambda_2 < 0$:
  - Two linearly independent eigenvectors v^1, v^2: the orbits are half-lines towards the origin *star node*
  - Only one eigenvector v: the orbits become parallel to the half-line defined by v *degenerate node*
- $\lambda_1 = \lambda_2 > 0$: Analogous, but orbits running in the opposite direction.

Complex eigenvalues

- $\lambda_{1,2}$ complex conjugate: $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$
- Complex solutions: $e^{(\alpha \pm \beta i)t}$
- Real solutions: Linear combinations of $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$
- Three cases:
  - $\alpha = 0$: Circle *Center*
  - $\alpha < 0$: Inward spiral *Stable focus*
  - $\alpha > 0$: Outward spiral *Unstable focus*