Focal sets

Consider the set $\ensuremath{\mathcal{D}}$ of phase space domains.

- define $R(D) := \{ D' \in \mathcal{D} \mid D' \text{ regular}, D \subseteq \partial D' \}$
- ▶ for $D \in \mathcal{D}$ the set $\Psi(D)$ is called (qualitative)*focal set* with
 - $\Psi(D) := \{\Phi(D)\}$, if D is regular
 - ▶ $\Psi(D) := C \cap \overline{rect}(\{\Phi(D') \mid D' \in R(D)\})$, if *D* is a switching domain, where *C* is the lowest dimensional affine subspace in \mathbb{R}^n containing *D* and $\overline{rect}(E)$ is the smallest closed paraxial hyperrectangle containing *E*

Remark: A better choice for the focal set of a switching domain would be the closed convex hull of the target values of the adjacent regular domains instead of the hyperrectangle, however, that choice might lead to complex polytopes and difficulties for computation.

Relative position vectors

the function v : D×Ω → {−1,0,1}ⁿ maps a domain D and a state x to the *relative position vector* v(D,x) where for all i ∈ {1,...,n}

$$v_i(D,x) := \begin{cases} 1 & \text{if for all } d \in D : x_i > d_i, \\ 0 & \text{if for some } d \in D : x_i = d_i, \\ -1 & \text{if for all } d \in D : x_i < d_i. \end{cases}$$

▶ for a set $M \subseteq \Omega$, $D \in \mathcal{D}$, we define $V(D, E) := \{v(D, x) \mid x \in M\}$

Remark: Focal sets and relative position vectors for domains and corresponding focal sets are the only ingredients needed to obtain a graph representation of the dynamics.

Qualitative state transition graph

Consider a directed graph Q with

- $\blacktriangleright \ \text{vertex set} \ \mathcal{D}$
- $D \rightarrow D'$ edge iff
 - ► $D' \in \partial D$, $\Psi(D) \neq \emptyset$ and there exists $v \in V(D, \Psi(D))$ with $v_i w_i = 1$ for all $i \in I' \setminus I$, or
 - ► $D \in \partial D'$, $\Psi(D') \neq \emptyset$ and there exists $v \in V(D', \Psi(D'))$ with $v_i w_i \neq -1$ for all $i \in I \setminus I'$

with $V(D, D') = \{w\}$, *I* switching variables of *D*, *I'* switching variables of *D'*

Qualitative state transition graph

Consider a directed graph Q with

- vertex set D
- $D \rightarrow D'$ edge iff
 - ► $D' \in \partial D$, $\Psi(D) \neq \emptyset$ and there exists $v \in V(D, \Psi(D))$ with $v_i w_i = 1$ for all $i \in I' \setminus I$, or
 - ► $D \in \partial D'$, $\Psi(D') \neq \emptyset$ and there exists $v \in V(D', \Psi(D'))$ with $v_i w_i \neq -1$ for all $i \in I \setminus I'$

with $V(D, D') = \{w\}$, *I* switching variables of *D*, *I'* switching variables of *D'*

Theorem Given a solution $\xi : [0, \tau] \to \Omega, \tau > 0$, of a PLDE model (1.3 (1)) passing through a (finite) sequence of domains (D^0, \ldots, D^m) in Ω , then (D^0, \ldots, D^m) is a path in the corresponding qualitative state transition graph Q. [Proof: see e.g. de Jong et al., 2004]

Parameter insensitivity

Given a PLDE model we consider

• threshold ordering for all $i \in \{1, ..., n\}$:

 $0 < \theta_i^1 < \cdots < \theta_i^{p_i} < max_i$ (threshold constraint)

► equilibrium ordering w.r.t. thresholds for all regular domains D: for all i ∈ {1,...,n} exactly one of the following inequalities holds

$$\begin{array}{l} 0 < \mu_i^D / \gamma_i < \theta_i^1, \\ \theta_i^k < \mu_i^D / \gamma_i < \theta_i^{k+1}, \ k \in \{1, \dots, p_i - 1\}, \\ \theta_i^{p_i} < \mu_i^D / \gamma_i < max_i \end{array}$$
(equilibrium constraint)

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$$\begin{split} &0 < \mu_i^D / \gamma_i < \theta_i^1, \\ &\theta_i^k < \mu_i^D / \gamma_i < \theta_i^{k+1}, \ k \in \{1, \dots, p_i - 1\}, \end{split} \tag{equilibrium constraint} \\ &\theta_i^{p_i} < \mu_i^D / \gamma_i < max_i \end{split}$$

Theorem All PLDE models defined by the same equations (1.3(1)) that satisfy the same threshold and equilibrium constraints give rise to the same qualitative state transition graph.

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[Proof: see e.g. de Jong et al., 2004]
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