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Algorithms

WS 2015/16

Exercise 5
(discussed on November 20th, 2015)

Prepare yourself to present your solutions to your fellow students.

1. **”’sparse”’ skip list (Niveau I)**

   Each node in the skip list has up to two incoming directed edges pointing to other nodes in the skip list.

   (a) Which edges are really necessary for a search and which can be removed?

   (b) Can you give a rough estimate for the expected number of edges that can be removed?

2. **Skip lists (Niveau II)**

   Proof that the height of a skip list has expected value $O(\log n)$ with high probability, i.e. show that the probability that the height deviates from $\log n$ by a large factor is very low.

   Hint: You do not need Chernoff bounds or Markov’s inequality to show this.

3. **Expected values (Niveau I)**

   Let X and Y be random variables:

   (a) Prove that $E(X + Y) = E(X) + E(Y)$.

   (b) Assume that X and Y are independent. Prove that $E(XY) = E(X)E(Y)$

   (c) Assume that X takes values $\{0, 1, 2, \ldots\}$. Show that $E(X) = \sum_{k=1}^{\infty} Pr(X \geq k)$.

4. **Tree decomposition (Niveau I)**

   Prove the following theorem:
   Let $G = (V, E)$ be a graph, T be a tree decomposition of G, and (x, y) an edge in T. The deletion of (x, y) divides T into two components X and Y. Let V_x and V_y be the ‘pieces’ of x and y, respectively. Then deleting the set $V_x \cap V_y$ from V disconnects G into the two subgraphs $G_X - (V_x \cap V_y)$ and $G_Y - (V_x \cap V_y)$.
   (G_M for $M = X, Y$ is the subgraph of G that consists of all nodes in the ‘pieces’ of M.)