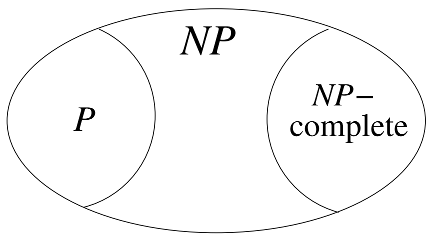
Polynomial reductions

- A *polynomial reduction* of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a polynomially computable function $f : \Sigma_1^* \to \Sigma_2^*$ with $w \in L_1 \Leftrightarrow f(w) \in L_2$.
- **Proposition.** If L_1 is polynomially reducible to L_2, then
  1. $L_1 \in P$ if $L_2 \in P$ and $L_1 \in NP$ if $L_2 \in NP$
  2. $L_2 \notin P$ if $L_1 \notin P$ and $L_2 \notin NP$ if $L_1 \notin NP$.
- L_1 and L_2 are *polynomially equivalent* if they are polynomially reducible to each other.

NP-complete problems

- A language $L \subseteq \Sigma^*$ is *NP-complete* if
  1. $L \in NP$
  2. Any $L' \in NP$ is polynomially reducible to L.
- **Proposition.** If L is *NP*-complete and $L \in P$, then $P = NP$.
- **Corollary.** If L is *NP*-complete and $P \neq NP$, then there exists no polynomial algorithm for L.

Structure of the class NP



Fundamental open problem: $P \neq NP$ **?**

Proving NP-completeness

- **Theorem** (Cook 1971). SAT is *NP*-complete.
- **Proposition.** L is *NP*-complete if
  1. $L \in NP$
  2. there exists an *NP*-complete problem L' that is polynomially reducible to L.
- *Example:* INDEPENDENT SET

  Instance: Graph $G = (V, E)$ and $k \in \mathbb{N}, k \leq |V|$.
  Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and no two vertices in V' are joined by an edge in E ?

Reducing 3SAT to INDEPENDENT SET

- Let F be a conjunction of n *clauses* of length 3, i.e., a disjunction of 3 propositional variables or their negation.
- Construct a graph G with $3n$ vertices that correspond to the variables in F.
- For any clause in F, connect by three edges the corresponding vertices in G.
- Connect all pairs of vertices corresponding to a variable x and its negation $\neg x$.
- F is satisfiable if and only if G contains an independent set of size n.

NP-hard problems

- *Decision problem:* solution is either yes or no
- Example: Traveling salesman decision problem:
  Given a network of cities, distances, and a number B, does there exist a tour with length $\leq$ B?
- *Search problem:* find an object with required properties
- Example: Traveling salesman optimization problem:
  Given a network of cities and distances, find a shortest tour.
- Decision problem *NP*-complete $\Rightarrow$ search problem *NP*-hard
- *NP-hard problems:* at least as hard as *NP*-complete problems

NP-hard problems in bioinformatics

Sperschneider 08

- Multiple alignment
- Shortest common superstring
- Protein threading
- Pseudoknot prediction
- Bi-Clustering
- ...

Further complexity classes

coNP:	Problems whose complement is in *NP*
PSPACE:	Problems solvable in polynomial space
EXPTIME:	Problems solvable in exponential time

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