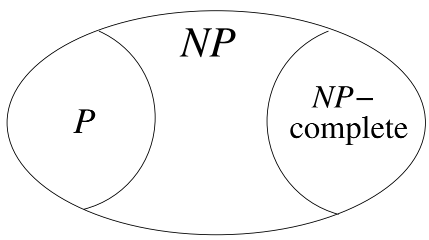
## Polynomial reductions

- A *polynomial reduction* of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a polynomially computable function $f : \Sigma_1^* \to \Sigma_2^*$ with $w \in L_1 \Leftrightarrow f(w) \in L_2$.
- **Proposition.** If $L_1$ is polynomially reducible to $L_2$, then
  1. $L_1 \in P$ if $L_2 \in P$ and $L_1 \in NP$ if $L_2 \in NP$
  2. $L_2 \notin P$ if $L_1 \notin P$ and $L_2 \notin NP$ if $L_1 \notin NP$.
- $L_1$ and $L_2$ are *polynomially equivalent* if they are polynomially reducible to each other.

## NP-complete problems

- A language $L \subseteq \Sigma^*$ is *NP-complete* if
  1. $L \in NP$
  2. Any $L' \in NP$ is polynomially reducible to $L$.
- **Proposition.** If $L$ is $NP$-complete and $L \in P$, then $P = NP$.
- **Corollary.** If $L$ is $NP$-complete and $P \neq NP$, then there exists no polynomial algorithm for $L$.

## Structure of the class NP




**Fundamental open problem:** $P \neq NP$ **?**

## Proving NP-completeness

- **Theorem** (Cook 1971). SAT is $NP$-complete.
- **Proposition.** $L$ is $NP$-complete if
  1. $L \in NP$
  2. there exists an $NP$-complete problem $L'$ that is polynomially reducible to $L$.
- *Example:* INDEPENDENT SET

  Instance: Graph $G = (V, E)$ and $k \in \mathbb{N}, k \leq |V|$.
  Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and no two vertices in $V'$ are joined by an edge in $E$ ?

## Reducing 3SAT to INDEPENDENT SET

- Let $F$ be a conjunction of $n$ *clauses* of length 3, i.e., a disjunction of 3 propositional variables or their negation.
- Construct a graph $G$ with $3n$ vertices that correspond to the variables in $F$.
- For any clause in $F$, connect by three edges the corresponding vertices in $G$.
- Connect all pairs of vertices corresponding to a variable $x$ and its negation $\neg x$.
- $F$ is satisfiable if and only if $G$ contains an independent set of size $n$.

## NP-hard problems

- *Decision problem:* solution is either yes or no
- Example: Traveling salesman decision problem:
  Given a network of cities, distances, and a number B, does there exist a tour with length $\leq$ B?
- *Search problem:* find an object with required properties
- Example: Traveling salesman optimization problem:
  Given a network of cities and distances, find a shortest tour.
- Decision problem *NP*-complete $\Rightarrow$ search problem *NP*-hard
- *NP-hard problems:* at least as hard as *NP*-complete problems

## NP-hard problems in bioinformatics

*Sperschneider 08*

- Multiple alignment
- Shortest common superstring
- Protein threading
- Pseudoknot prediction
- Bi-Clustering
- ...

## Further complexity classes

| | |
|---|---|
| *coNP:* | Problems whose complement is in *NP* |
| *PSPACE:* | Problems solvable in polynomial space |
| *EXPTIME:* | Problems solvable in exponential time |
| ⋮ | |

## Literature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