Bifurcations

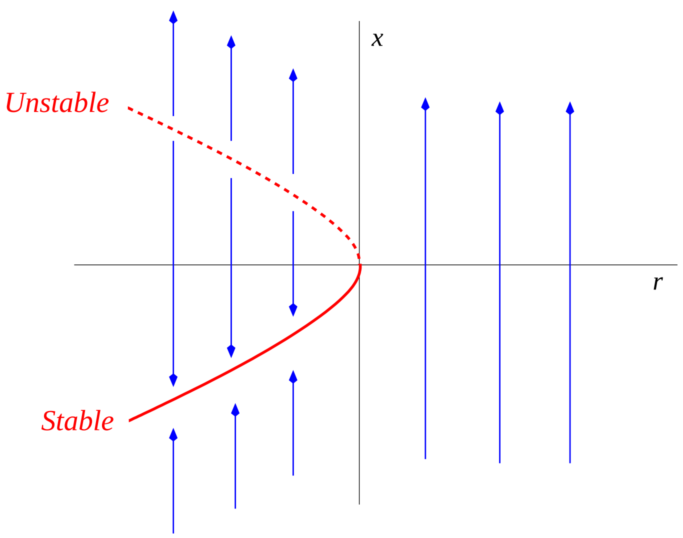
- The qualitative behavior of the solutions of a system of differential equations depends on the parameters.
- Qualitative changes in the dynamics are called *bifurcations*, and the parameter values at which they occur are called *bifurcation points.*
- Bifurcation of equilibrium solutions
  - A negative real eigenvalue becomes positive.
  - A negative real part of a pair of conjugate complex eigenvalues becomes positive.

Saddle-node bifurcation

- Creation and destruction of critical points
- *Prototypical example* (in dimension 1): $\dot{x} = r + x^2$, with a parameter $r \in \mathbb{R}$.
- Three cases:
  1. $r < 0$: Two critical points, one stable and one unstable.
  2. $r = 0$: One half-stable critical point.
  3. $r > 0$. No critical point.

  $\rightsquigarrow$ bifurcation at $r = 0$
- Locally, all saddle node bifurcations have the **normal form** $\dot{x} = r \pm x^2$, with $r \in \mathbb{R}$.

Bifurcation diagram

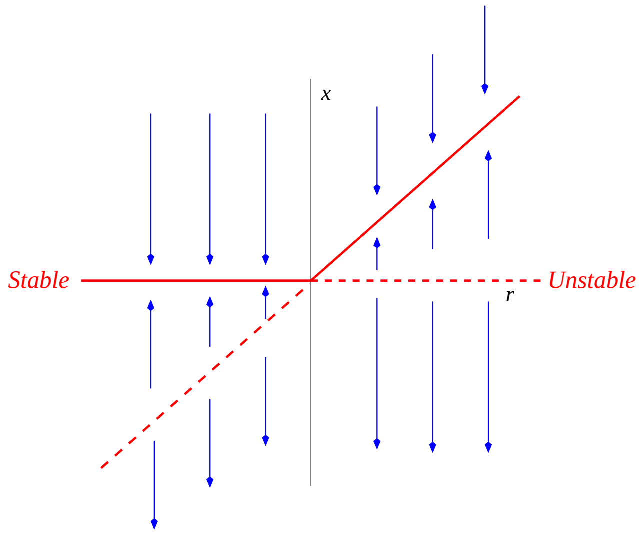


Transcritical bifurcation

- Change in stability of a critical point
- *Normal form:* $\dot{x} = rx - x^2$, with a parameter $r \in \mathbb{R}$.
- Three cases
  1. $r < 0$: Two critical points, an unstable one at $x^* = r$ and a stable one at $x^* = 0$.
  2. $r = 0$: One half-stable critical point.
  3. $r > 0$: Two critical points, a stable one at $x^* = r$ and an unstable one at $x^* = 0$.

  $\rightsquigarrow$ bifurcation at $r = 0$

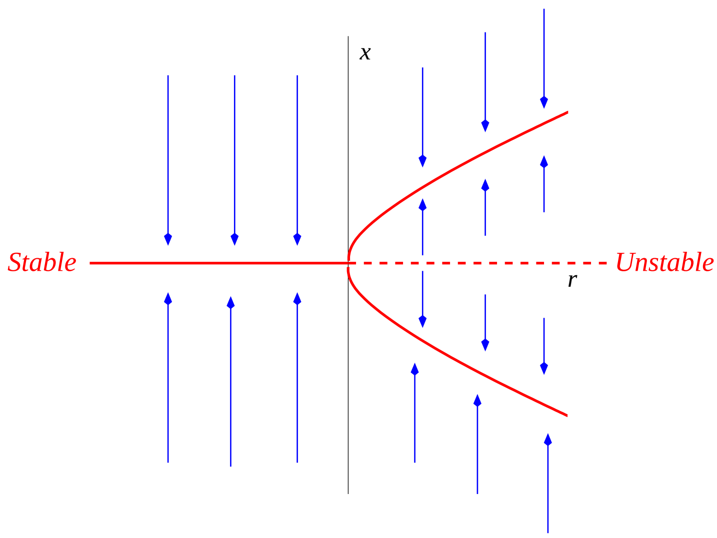
Bifurcation diagram



Pitchfork bifurcation

- Common in problems having a symmetry, e.g. buckling of a beam.
- *Supercritical pitchfork bifurcation:* $\dot{x} = rx - x^3$
  1. $r < 0$: One stable critical point at $x^* = 0$.
  2. $r = 0$: Still one stable critical point at $x^* = 0$, but much slower decay ("critical slow down").
  3. $r > 0$: Origin becomes unstable, two new stable critical points appear, symmetrically located at $x^* = \pm\sqrt{r}$.
- *Subcritical pitchfork bifurcation:* $\dot{x} = rx + x^3$
  1. $r < 0$: Two unstable critical points at $x^* = \pm\sqrt{r}$, stable critical point at $x^* = 0$.
  2. $r > 0$: Unstable critical point at $x^* = 0$, with $x(t) \to \pm\infty$, for any initial condition $x(0) \neq 0 \rightsquigarrow$ "blow-up" (x^3 term is destabilizing).

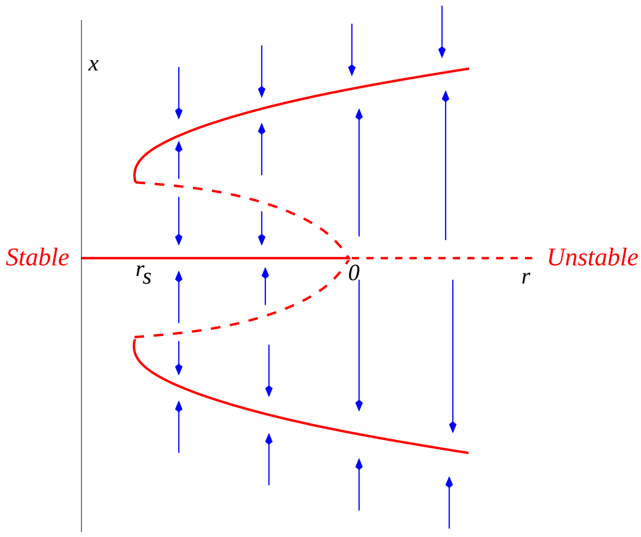
Bifurcation diagram



Stabilizing and destabilizing influence

- In real systems, the explosive instability of a subcritical pitchfork bifurcation is usually opposed by the stabilizing influence of higher-order terms.
- *Example:* $\dot{x} = rx + x^3 - x^5, r \in \mathbb{R}$.
- For small x, the diagram looks like the subcritical pitchfork bifurcation $\dot{x} = rx + x^3$.
- However, due to the x^5 term, the unstable branches turn around and become stable at $r = r_s < 0$.
- The bifurcation at $r = r_s$ is a saddle-node bifurcation.
- Stable large-amplitude branches exist for all $r > r_s$.

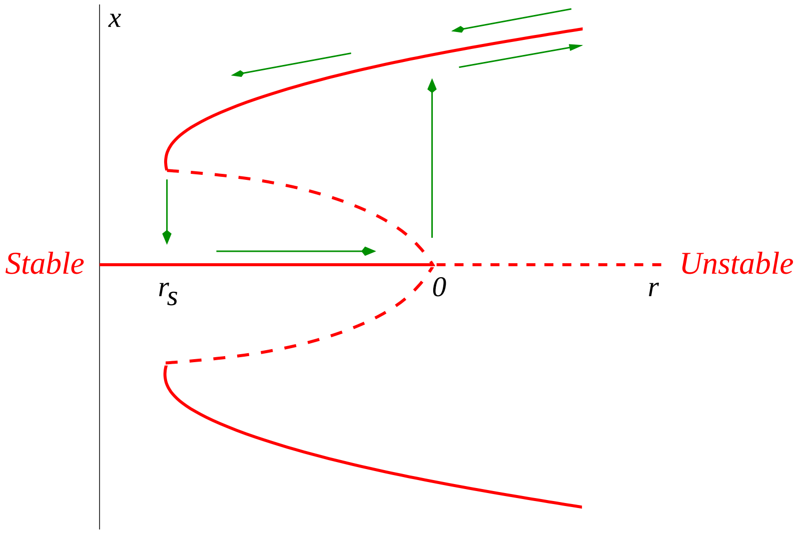
Bifurcation diagram



Jumps and hysteresis

- For $r_s < r < 0$, there are three stable critical points. The initial value $x(0)$ determines which one is approached as $t \to \infty$. The origin is locally stable, but not globally stable.
- If we start in the state $x^* = 0$ and increase the parameter $r < 0$, the state remains at the origin until $r = 0$.
- At $r = 0$, the origin loses stability. Now the slightest perturbation causes the state to *jump* to one of the large-amplitude branches.
- If r is further increased, the state moves along the large-amplitude branch.
- If r is decreased at some point, the state remains on the large-amplitude branch, even when r is decreased below 0.
- We have to lower r down past r_s to get the state jump back to the origin.
- The lack of reversibility as a parameter varies is called *hysteresis.*

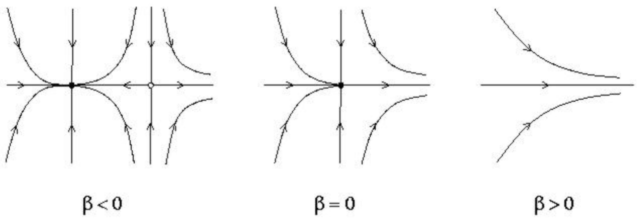
Bifurcation diagram



Higher dimensions

- Saddle-node, transcritical, and pitchfork bifurcations exist also in higher dimensions.
- *Example:* Saddle-node bifurcation

$$\begin{aligned}\dot{x} &= \beta + x^2 \\ \dot{y} &= -y\end{aligned}$$



- But, for $n \geq 2$, new phenomena occur $\rightsquigarrow$ Hopf bifurcations

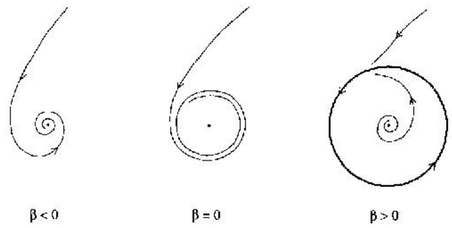
Supercritical Hopf bifurcation

- *Example:* 2-dimensional system in polar coordinates (r, θ)

$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + br^2\end{aligned}$$

  with parameters $\mu, \omega, b \in \mathbb{R}$.

- Three cases
  - $\mu < 0$: The origin $r = 0$ is a stable spiral, whose sense of rotation depends on the sign of ω.
  - $\mu = 0$: The origin is still a stable spiral with very slow decay.
  - $\mu > 0$: Unstable spiral at the origin and a stable circular limit cycle at $r = \sqrt{\mu}$.

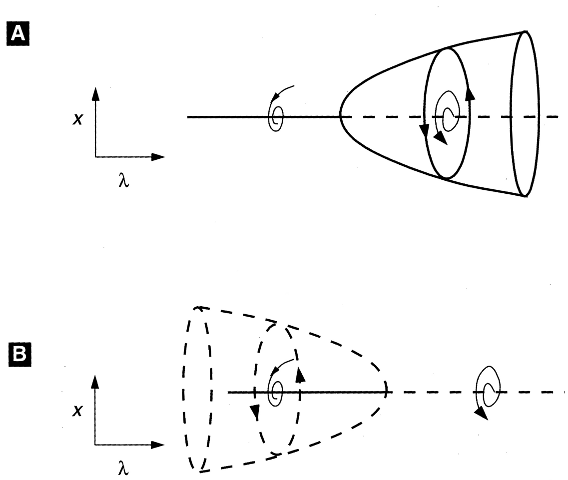


- Cartesian coordinates: $x = r\cos\theta, y = r\sin\theta$

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta = \mu x - \omega y + \text{ cubic terms} \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta = \omega x + \mu y + \text{ cubic terms} \end{aligned}$$

- Jacobi matrix at the origin: $A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$
- The eigenvalues $\lambda = \mu \pm \omega i$ cross the imaginary axis from left to right as μ increases from negative to positive values.

Hopf bifurcations



Subcritical Hopf bifurcation

- *Example:* 2-dimensional system in polar coordinates (r, θ)

$$\begin{aligned} \dot{r} &= \mu r + r^3 - r^5 \\ \dot{\theta} &= \omega + br^2 \end{aligned}$$

- Three cases
  - $\mu < 0$: Stable limit cycle and a stable critical point at the origin, with an unstable cycle in between. As μ increases, the unstable cycle tightens around the critical point.
  - $\mu = 0$: The cycle shrinks to 0 amplitude and engulfs the origin, rendering it unstable.
  - $\mu > 0$: The large-amplitude limit cycle is the only attractor. Solutions that used to remain near the origin are now forced to grow into large-amplitude oscillations.
- The systems exhibits hysteresis: once large-amplitude oscillations have begun, they cannot be turned off by bringing μ back to 0.

Limit cycles

- A *limit cycle* is an isolated closed trajectory. *Isolated* means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.
- **Poincaré-Bendixson theorem.** Suppose
  - R is a closed bounded subset of the plane;
  - $\dot{x} = f(x)$ is continuously differentiable on an open subset containing R;
  - R does not contain any critical points; and
  - there exists a trajectory C starting in R and staying in R for all future time.

  Then either C is a closed orbit, or it spirals towards a closed orbit for $t \to \infty$. In either case, R contains a closed orbit.

Lorenz equations

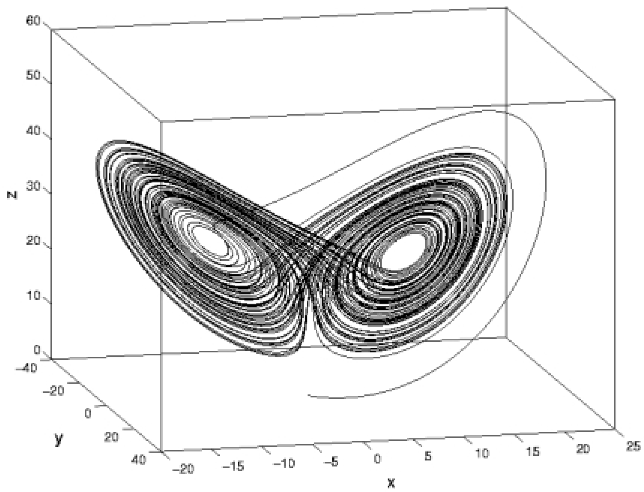
- *Lorenz equations*

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \text{ with parameters } \sigma, r, b > 0. \end{aligned}$$

  $\rightsquigarrow$ *chaotic behavior*

- *Chaos:* Aperiodic behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Lorenz trajectory



$\rightsquigarrow$ *strange attractor*