Bifurcations

- The qualitative behavior of the solutions of a system of differential equations depends on the parameters.
- Qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points.
- Bifurcation of equilibrium solutions
  - A negative real eigenvalue becomes positive.
  - A negative real part of a pair of conjugate complex eigenvalues becomes positive.

Saddle-node bifurcation

- Creation and destruction of critical points
- Prototypical example (in dimension 1): \( \dot{x} = r + x^2 \), with a parameter \( r \in \mathbb{R} \).
- Three cases:
  1. \( r < 0 \): Two critical points, one stable and one unstable.
  2. \( r = 0 \): One half-stable critical point.
  3. \( r > 0 \): No critical point.

\( \Rightarrow \) bifurcation at \( r = 0 \)

- Locally, all saddle node bifurcations have the normal form \( \dot{x} = r \pm x^2 \), with \( r \in \mathbb{R} \).

Bifurcation diagram

Transcritical bifurcation

- Change in stability of a critical point
- Normal form: \( \dot{x} = rx - x^2 \), with a parameter \( r \in \mathbb{R} \).
- Three cases
  1. \( r < 0 \): Two critical points, an unstable one at \( x^* = r \) and a stable one at \( x^* = 0 \).
  2. \( r = 0 \): One half-stable critical point.
  3. \( r > 0 \): Two critical points, a stable one at \( x^* = 0 \) and an unstable one at \( x^* = r \).

\( \Rightarrow \) bifurcation at \( r = 0 \)
**Pitchfork bifurcation**

- Common in problems having a symmetry, e.g. buckling of a beam.

- **Supercritical pitchfork bifurcation:** $\dot{x} = rx - x^3$
  1. $r < 0$: One stable critical point at $x^* = 0$.
  2. $r = 0$: Still one stable critical point at $x^* = 0$, but much slower decay (“critical slow down”).
  3. $r > 0$: Origin becomes unstable, two new stable critical points appear, symmetrically located at $x^* = \pm \sqrt{r}$.

- **Subcritical pitchfork bifurcation:** $\dot{x} = rx + x^3$
  1. $r < 0$: Two unstable critical points at $x^* = \pm \sqrt{r}$, stable critical point at $x^* = 0$.
  2. $r > 0$: Unstable critical point at $x^* = 0$, with $x(t) \to \pm \infty$, for any initial condition $x(0) \neq 0 \to “\text{blow-up}”$ ($x^3$ term is destabilizing).
Stabilizing and destabilizing influence

- In real systems, the explosive instability of a subcritical pitchfork bifurcation is usually opposed by the stabilizing influence of higher-order terms.

- Example: $\dot{x} = rx + x^3 - x^5, r \in \mathbb{R}$.

- For small $x$, the diagram looks like the subcritical pitchfork bifurcation $\dot{x} = rx + x^3$.

- However, due to the $x^5$ term, the unstable branches turn around and become stable at $r = r_s < 0$.

- The bifurcation at $r = r_s$ is a saddle-node bifurcation.

- Stable large-amplitude branches exist for all $r > r_s$.

Bifurcation diagram

Jumps and hysteresis

- For $r_s < r < 0$, there are three stable critical points. The initial value $x(0)$ determines which one is approached as $t \to \infty$. The origin is locally stable, but not globally stable.

- If we start in the state $x^* = 0$ and increase the parameter $r < 0$, the state remains at the origin until $r = 0$.

- At $r = 0$, the origin loses stability. Now the slightest perturbation causes the state to jump to one of the large-amplitude branches.

- If $r$ is further increased, the state moves along the large-amplitude branch.

- If $r$ is decreased at some point, the state remains on the large-amplitude branch, even when $r$ is decreased below 0.

- We have to lower $r$ down past $r_s$ to get the state jump back to the origin.

- The lack of reversibility as a parameter varies is called hysteresis.
Higher dimensions

- Saddle-node, transcritical, and pitchfork bifurcations exist also in higher dimensions.
- Example: Saddle-node bifurcation

\[
\begin{align*}
\dot{x} &= \beta + x^2 \\
\dot{y} &= -y
\end{align*}
\]

- But, for \( n \geq 2 \), new phenomena occur \( \iff \) Hopf bifurcations

**Supercritical Hopf bifurcation**

- Example: 2-dimensional system in polar coordinates \((r, \theta)\)

\[
\begin{align*}
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= \omega + br^2
\end{align*}
\]

with parameters \( \mu, b \in \mathbb{R} \).

- Three cases
  - \( \mu < 0 \): The origin \( r = 0 \) is a stable spiral, whose sense of rotation depends on the sign of \( \omega \).
  - \( \mu = 0 \): The origin is still a stable spiral with very slow decay.
  - \( \mu > 0 \): Unstable spiral at the origin and a stable circular limit cycle at \( r = \sqrt{\mu} \).
• Cartesian coordinates: $x = r \cos \theta, y = r \sin \theta$

$$
\dot{x} = r \cos \theta - r \dot{\theta} \sin \theta = \mu x - \omega y + \text{cubic terms}
$$

$$
\dot{y} = r \sin \theta + r \dot{\theta} \cos \theta = \omega x + \mu y + \text{cubic terms}
$$

• Jacobi matrix at the origin: $A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$

• The eigenvalues $\lambda = \mu \pm \omega i$ cross the imaginary axis from left to right as $\mu$ increases from negative to positive values.

**Hopf bifurcations**

**Subcritical Hopf bifurcation**

• Example: 2-dimensional system in polar coordinates $(r, \theta)$

$$
\dot{r} = \mu r + r^3 - r^5
$$

$$
\dot{\theta} = \omega + br^2
$$

• Three cases
  - $\mu < 0$: Stable limit cycle and a stable critical point at the origin, with an unstable cycle in between. As $\mu$ increases, the unstable cycle tightens around the critical point.
  - $\mu = 0$: The cycle shrinks to 0 amplitude and engulfs the origin, rendering it unstable.
  - $\mu > 0$: The large-amplitude limit cycle is the only attractor. Solutions that used to remain near the origin are now forced to grow into large-amplitude oscillations.

• The systems exhibit hysteresis: once large-amplitude oscillations have begun, they cannot be turned off by bringing $\mu$ back to 0.
Limit cycles

- A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.

- **Poincaré-Bendixson theorem.** Suppose
  - $R$ is a closed bounded subset of the plane;
  - $\dot{x} = f(x)$ is continuously differentiable on an open subset containing $R$;
  - $R$ does not contain any critical points; and
  - there exists a trajectory $C$ starting in $R$ and staying in $R$ for all future time.

Then either $C$ is a closed orbit, or it spirals towards a closed orbit for $t \to \infty$. In either case, $R$ contains a closed orbit.

Lorenz equations

- **Lorenz equations**

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz,
\end{align*}
\]

with parameters $\sigma, r, b > 0$.

\(\Rightarrow\) chaotic behavior

- **Chaos:** Aperiodic behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

Lorenz trajectory

\(\Rightarrow\) strange attractor