Reminder: Eigenvalues and eigenvectors

- Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $A \in \mathbb{F}^{n\times n}$ is a matrix.
- $\lambda \in \mathbb{F}$ is called an *eigenvalue* of A if there exists an *eigenvector* $v \in \mathbb{F}^n, v \neq 0$ such that $Av = \lambda v$.
- *Characteristic polynomial*

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

$\rightsquigarrow$ polynomial of degree n

- **Lemma.** λ is an eigenvalue of A if and only if $\chi_A(\lambda) = 0$.

Stability analysis of two-dimensional linear systems

- $\dot{y} = Ay$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ non-singular
- Characteristic equation:

$$\det(A - \lambda I) = \lambda^2 - (\underbrace{a+d}_{S})\lambda + (\underbrace{ad-bc}_{P}) = 0$$

- Eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(S \pm \sqrt{S^2 - 4P})$$

- Different cases depending on whether or not
  - λ_1, λ_2 are real or complex,
  - λ_1, λ_2 (resp. their real part) are positive or negative.

Possible cases

- $S^2 - 4P \geq 0$: Real roots λ_1, λ_2
  - $\lambda_1 \cdot \lambda_2 > 0$: Node
  - $\lambda_1 \cdot \lambda_2 < 0$: Saddle point
- $S^2 - 4P < 0$: Complex conjugate roots $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$
  - $\alpha \neq 0$: Focus
  - $\alpha = 0$: Center

Real eigenvalues $\lambda_1 \neq \lambda_2$

- Diagonalization $\dot{z} = Dz$, with $D = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- Solutions: $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$
- In phase plane: $|z_2| = c|z_1|^{\lambda_2/\lambda_1}$

- Case $\lambda_2/\lambda_1 > 0$: parabolic orbits *Node*
  - $\lambda_1, \lambda_2 < 0$: *stable*
  - $\lambda_1, \lambda_2 > 0$: *unstable*
- Case $\lambda_2/\lambda_1 < 0$: hyperbolic orbits *Saddle point*

Real eigenvalues $\lambda_1 = \lambda_2$

- $\lambda_1 = \lambda_2 < 0$:
  - Two linearly independent eigenvectors v^1, v^2: the orbits are half-lines towards the origin *star node*
  - Only one eigenvector v: the orbits become parallel to the half-line defined by v *degenerate node*
- $\lambda_1 = \lambda_2 > 0$: Analogous, but orbits running in the opposite direction.

Complex eigenvalues

- $\lambda_{1,2}$ complex conjugate: $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$
- Complex solutions: $e^{(\alpha \pm \beta i)t}$
- Real solutions: Linear combinations of $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$
- Three cases:
  - $\alpha = 0$: Circle *Center*
  - $\alpha < 0$: Inward spiral *Stable focus*
  - $\alpha > 0$: Outward spiral *Unstable focus*

Differential equations: Qualitative theory

System of differential equations: $\dot{x} = f(t, x)$ or $\dot{x} = f(x)$

Basic questions

1. Do there exist equilibrium solutions $x(t) = a$?
2. Let $x(t), \tilde{x}(t)$ be two solutions with $\tilde{x}(0)$ close to $x(0)$.
   Will $\tilde{x}(t)$ remain close to $x(t)$ for all future time, or will $\tilde{x}(t)$ diverge from $x(t)$ as t approaches infinity ? *Stability*
3. What happens to solutions $x(t)$ as t approaches infinity?
   Do all solutions approach equilibrium values? If not, do they at least approach a periodic solution?

Stability of solutions

- The solution $x(t)$ of $\dot{x} = f(x)$ is *stable* if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ we have $||x(t) - \tilde{x}(t)|| < \varepsilon$, for all $t > 0$.
- The solution $x(t)$ of $\dot{x} = f(x)$ is *unstable* if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists a solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ but $||x(t) - \tilde{x}(t)|| \geq \varepsilon$, for some $t > 0$.
- $||a|| = ||a||_\infty = \max\{|a_1|, \ldots, |a_n|\}$ denotes the (maximum) *norm* of the vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

Stability of linear systems

- Consider a linear system
$$\dot{x} = Ax, \text{ with } A \in \mathbb{R}^{n \times n} \tag{*}$$

- **Theorem** (cf. Braun, Differential equations, Chapt. 4.2)
  - Every solution $x(t)$ of (*) is stable if all eigenvalues of A have negative real part < 0.
  - Every solution $x(t)$ of (*) is unstable if at least one eigenvalue of A has positive real part > 0.
  - If all eigenvalues of A have real part ≤ 0 and $\lambda_1, \dots, \lambda_s$ have zero real part, let k_j be the multiplicity of λ_j.
    Every solution $x(t)$ of (*) is stable if A has k_j linearly independent eigenvectors for each eigenvalue $\lambda_j, j = 1, \dots, s$. Otherwise, every solution $x(t)$ of (*) is unstable.

Reminder: Linear approximation

- One dimensional case
  - $f : I \to \mathbb{R}$ differentiable, $I \subseteq \mathbb{R}$ interval, $a \in I$
  - Linear approximation
$$f(x) = f(a) + f'(a) \cdot (x - a) + \text{higher order terms}$$
  - Taylor expansion
$$\begin{aligned} f(x) &= f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots \\ &\quad + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots \end{aligned}$$

- n-dimensional case
  - $f : D \to \mathbb{R}^n$ differentiable, $a \in D \subseteq \mathbb{R}^n$
  - Jacobi matrix
$$\frac{\partial f}{\partial x}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \dots & \frac{\partial f_n}{\partial x_n}(a) \end{pmatrix}$$
  - Linear approximation
$$f(x) = f(a) + \frac{\partial f}{\partial x}(a)(x-a) + \text{higher order terms}$$

Linearisation around a critical point

- $x = a$ critical point of $\dot{x} = f(x)$
- Linearisation
$$\dot{x} = \frac{\partial f}{\partial x}(a)(x-a) + \text{higher order terms}$$
- Study linear equation with constant coefficients
$$\dot{z} = \frac{\partial f}{\partial z}(a)(z-a)$$

- Shift point a to the origin by putting $y = z - a$ and $A = \frac{\partial f}{\partial x}(a)$.
- Linearised system

$$\dot{y} = Ay$$

Asymptotic stability

- Consider the autonomous system $\dot{x} = f(x)$.
- A solution $x(t)$ is *asymptotically stable* if it is stable and if every solution $\tilde{x}(t)$ which starts sufficiently close to $x(t)$ must approach $x(t)$ as t approaches infinity.

Stability of equilibrium solutions

Theorem (cf. Braun, Differential equations, Chapt. 4.3)

- Consider the system $\dot{x} = f(x)$ and suppose f has continuous second-order partial derivatives.
- Let $x(t) = a$ be an equilibrium solution and $A = \frac{\partial f}{\partial x}(a)$.
- The equilibrium solution $x(t) = a$ is asymptotically stable if all the eigenvalues of A have negative real part.
- The equilibrium solution $x(t) = a$ is unstable if at least one eigenvalue of A has positive real part.
- The stability of the equilibrium solution $x(t) = a$ cannot be determined from the stability of the equilibrium solution $y(t) = 0$ of the linear system $\dot{y} = Ay$ if all the eigenvalues of A have negative real part ≤ 0 and at least one eigenvalue of A has zero real part.