Reminder: Eigenvalues and eigenvectors

- Suppose \( F = \mathbb{R} \) or \( F = \mathbb{C} \) and \( A \in F^{n \times n} \) is a matrix.
- \( \lambda \in F \) is called an eigenvalue of \( A \) if there exists an eigenvector \( v \in \mathbb{F}^n, v \neq 0 \) such that \( Av = \lambda v \).

**Characteristic polynomial**

\[
\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}
\]

\[\Rightarrow \text{polynomial of degree } n\]

**Lemma.** \( \lambda \) is an eigenvalue of \( A \) if and only if \( \chi_A(\lambda) = 0 \).

**Stability analysis of two-dimensional linear systems**

- \( \dot{y} = Ay \) with \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) non-singular
- Characteristic equation:
  \[
  \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + (ad - bc) = 0
  \]
- Eigenvalues
  \[
  \lambda_{1,2} = \frac{1}{2}(S \pm \sqrt{S^2 - 4P})
  \]
- Different cases depending on whether or not
  - \( \lambda_1, \lambda_2 \) are real or complex,
  - \( \lambda_1, \lambda_2 \) (resp. their real part) are positive or negative.

**Possible cases**

- \( S^2 - 4P \geq 0 \): Real roots \( \lambda_1, \lambda_2 \)
  - \( \lambda_1 \cdot \lambda_2 > 0 \): Node
  - \( \lambda_1 \cdot \lambda_2 < 0 \): Saddle point
- \( S^2 - 4P < 0 \): Complex conjugate roots \( \lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0 \)
  - \( \alpha \neq 0 \): Focus
  - \( \alpha = 0 \): Center

**Real eigenvalues** \( \lambda_1 \neq \lambda_2 \)

- Diagonalization \( \dot{z} = Dz \), with \( D = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \)
- Solutions: \( z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} \)
- In phase plane: \( |z_2| = c|z_1|^{|\lambda_2/\lambda_1|} \)
• Case $\lambda_2/\lambda_1 > 0$: parabolic orbits
  - $\lambda_1, \lambda_2 < 0$: stable node
  - $\lambda_1, \lambda_2 > 0$: unstable node
• Case $\lambda_2/\lambda_1 < 0$: hyperbolic orbits

Real eigenvalues $\lambda_1 = \lambda_2$

• $\lambda_1 = \lambda_2 < 0$:
  - Two linearly independent eigenvectors $v^1, v^2$: the orbits are half-lines towards the origin star node
  - Only one eigenvector $v$: the orbits become parallel to the half-line defined by $v$ degenerate node
• $\lambda_1 = \lambda_2 > 0$: Analogous, but orbits running in the opposite direction.

Complex eigenvalues

• $\lambda_{1,2}$ complex conjugate: $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$
• Complex solutions: $e^{(\alpha \pm i\beta)t}$
• Real solutions: Linear combinations of $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$
• Three cases:
  - $\alpha = 0$: Circle Center
  - $\alpha < 0$: Inward spiral Stable focus
  - $\alpha > 0$: Outward spiral Unstable focus

Differential equations: Qualitative theory

System of differential equations: $\dot{x} = f(t, x)$ or $\dot{x} = f(x)$

Basic questions

1. Do there exist equilibrium solutions $x(t) = a$?

2. Let $x(t), \tilde{x}(t)$ be two solutions with $\tilde{x}(0)$ close to $x(0)$. Will $\tilde{x}(t)$ remain close to $x(t)$ for all future time, or will $\tilde{x}(t)$ diverge from $x(t)$ as $t$ approaches infinity? Stability

3. What happens to solutions $x(t)$ as $t$ approaches infinity?
  Do all solutions approach equilibrium values? If not, do they at least approach a periodic solution?

Stability of solutions

• The solution $x(t)$ of $\dot{x} = f(x)$ is stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ we have $||x(t) - \tilde{x}(t)|| < \varepsilon$, for all $t > 0$.

• The solution $x(t)$ of $\dot{x} = f(x)$ is unstable if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists a solution $\tilde{x}(t)$ with $||x(0) - \tilde{x}(0)|| < \delta$ but $||x(t) - \tilde{x}(t)|| \geq \varepsilon$, for some $t > 0$.

• $||a|| = ||a||_\infty = \max\{|a_1|, \ldots, |a_n|\}$ denotes the (maximum) norm of the vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. 
**Stability of linear systems**

- Consider a linear system
  \[ \dot{x} = Ax, \quad \text{with } A \in \mathbb{R}^{n \times n} \]  
  \(^(*)\)

- **Theorem** (cf. Braun, Differential equations, Chapt. 4.2)
  - Every solution \( x(t) \) of \(^(*)\) is stable if all eigenvalues of \( A \) have negative real part \(<0\).
  - Every solution \( x(t) \) of \(^(*)\) is unstable if at least one eigenvalue of \( A \) has positive real part \(>0\).
  - If all eigenvalues of \( A \) have real part \(\leq0\) and \(\lambda_1, \ldots, \lambda_s \) have zero real part, let \(k_j\) be the multiplicity of \(\lambda_j\).
    Every solution \( x(t) \) of \(^(*)\) is stable if \(A\) has \(k_j\) linearly independent eigenvectors for each eigenvalue \(\lambda_j, j = 1, \ldots, s\). Otherwise, every solution \( x(t) \) of \(^(*)\) is unstable.

**Reminder: Linear approximation**

- One dimensional case
  - \( f : I \to \mathbb{R} \) differentiable, \( I \subseteq \mathbb{R} \) interval, \( a \in I \)
  - Linear approximation
    \[ f(x) = f(a) + f'(a) \cdot (x - a) + \text{higher order terms} \]
  - Taylor expansion
    \[ f(x) = f(a) + f'(a) \cdot (x - a) + \frac{f''(a)}{2} (x - a)^2 + \ldots \]
    \[ + \frac{f^{(k)}(a)}{k!} (x - a)^k + \ldots \]

- \( n \)-dimensional case
  - \( f : D \to \mathbb{R}^n \) differentiable, \( a \in D \subseteq \mathbb{R}^n \)
  - Jacobi matrix
    \[ \frac{\partial f}{\partial x}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \ldots & \frac{\partial f_n}{\partial x_n}(a) \end{pmatrix} \]
  - Linear approximation
    \[ f(x) = f(a) + \frac{\partial f}{\partial x}(a)(x - a) + \text{higher order terms} \]

**Linearisation around a critical point**

- \( x = a \) critical point of \( \dot{x} = f(x) \)
  - Linearisation
    \[ \dot{x} = \frac{\partial f}{\partial x}(a)(x - a) + \text{higher order terms} \]
  - Study linear equation with constant coefficients
    \[ \dot{z} = \frac{\partial f}{\partial z}(a)(z - a) \]
• Shift point $a$ to the origin by putting $y = z - a$ and $A = \frac{\partial f}{\partial x}(a)$.

• Linearised system

$$\dot{y} = Ay$$

**Asymptotic stability**

• Consider the autonomous system $\dot{x} = f(x)$.

• A solution $x(t)$ is \textit{asymptotically stable} if it is stable and if every solution $\tilde{x}(t)$ which starts sufficiently close to $x(t)$ must approach $x(t)$ as $t$ approaches infinity.

**Stability of equilibrium solutions**

\textbf{Theorem} (cf. Braun, Differential equations, Chapt. 4.3)

• Consider the system $\dot{x} = f(x)$ and suppose $f$ has continuous second-order partial derivatives.

• Let $x(t) = a$ be an equilibrium solution and $A = \frac{\partial f}{\partial x}(a)$.

• The equilibrium solution $x(t) = a$ is asymptotically stable if all the eigenvalues of $A$ have negative real part.

• The equilibrium solution $x(t) = a$ is unstable if at least one eigenvalue of $A$ has positive real part.

• The stability of the equilibrium solution $x(t) = a$ cannot be determined from the stability of the equilibrium solution $y(t) = 0$ of the linear system $\dot{y} = Ay$ if all the eigenvalues of $A$ have negative real part $\leq 0$ and at least one eigenvalue of $A$ has zero real part.