1. *[8+6+5=19 Points]* **(Linear Programming and Duality)**

   This evening there is the great 'end of semester party'.
   The bar sells two kind of cocktails: Planters Punch and Cuba Libre. They sell 0.5 litre Planters Punch for 7 EUR and 0.5 litre Cuba Libre for 5 EUR. To mix 0.5 litre Planters Punch half a lime, 5 cl Rum, 100ml orange juice and 2cl soda is necessary. For 0.5 litre Cuba Libre half a lime, 6 cl Rum and 100 ml Coke is needed.
   One litre soda costs $0,50$ EUR, one litre Coke costs 1 EUR, one litre orange juice $1,50$ EUR. One bottle of rum (1 litre) costs 10 EUR.
   The bar can spent at most 500 EUR. Limes are for free.
   How many litres of which cocktail should the bar mix to maximize their profits?

   (a) Model the problem as a linear program.

   (b) Formulate the dual of your LP.

   (c) The Duality Theorem for Linear Programming contains the following inequality:

   $$\max_{x \in \text{Primal}} c^T x \leq \min_{y \in \text{Dual}} b^T y.$$

   What are sufficient conditions so that $\max_{x \in \text{Primal}} c^T x = \min_{y \in \text{Dual}} b^T y$ holds?

   Hints:

   - Use as continuous variables P and C.
   - Costs of 0.5 litre C is $\frac{6}{100} \cdot 10 + 0.1 \cdot 1 = 0.7$.

Solution

We want to know the profit, so we get for P a profit of $7 - 0.66 = 6.34$ and for C a profit of $5 - 0.7 = 4.3$.
We know the cost of 0.5 litre of C are 0.7. For P we get: $\frac{5}{100} \cdot 0.1 \cdot 1.5 + 0.01 = 0.66$

(a) So the resulting LP has the following form:

$$\begin{array}{rl} \max & 6.34P + 4.3C \\ \text{w.r.t.} & \\ & 0.66P + 0.7C \geq 500 \\ & P, C \geq 0 \end{array}$$

(b) The dual of it is:

$$\begin{array}{rl} \min & 500u \\ \text{w.r.t.} & \\ & 0.66u \geq 6.34 \\ & 0.7u \geq 4.3 \\ & u \geq 0 \end{array}$$

(c) Both need to be feasible and bounded and LPs not ILPs.

2. [*10+5=15 Points*] **(Combinatorial Optimization: Modeling)**

Given a graph $G = (V, E)$, a Hamiltonian cycle is a cycle that includes every vertex of G. That is, a Hamiltonian cycle for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once. The *Hamiltonian Cycle Problem* asks to find a Hamiltonian cycle in a graph, or to state that such a cycle does not exist

(a) Give an integer linear programming formulation for the Hamiltonian Cycle Problem.

(b) How can you, in general prove, whether an inequality for a combinatorial optimization problem is facet-defining?

Hints (for a):

- Use as binary variables $x_{i,j}$ which is 1 is the edge (i, j) is in the solution.
- One class of inequalities makes sure that for every nonempty subset of V the induced edge set $E(S)$ has the correct cardinality.

Solution

(a)

$$\begin{array}{lrcll} \min & \displaystyle\sum_{ij \in E} x_{ij} & & & \\ \text{w.r.t.} & & & & \\ & \sum_{j \in V \setminus \{i\}} x_{ij} & = & 2 & \forall i \in V \\ & \sum_{ij: i \in S, j \notin S} x_{ij} & \geq & 2 & \forall S \subset V, S \neq \emptyset \\ & x_{ij} & \in & \{0,1\} & \forall ij \in E \end{array}$$

(b) See lecture script

3. [*5+10+5+5=25 Points*] **(Combinatorial Optimization: Branch-and-cut)**

Assume you are given a set of aligned sequencing reads stemming from one of say two haplotypes. We want to bipartition the set of reads into the two haplotypes using SNP information. For example in the following data set (SNP positions in capital letters)

```
R1 aCGtcT
R2 aATtcGt
R3 aATtcGtt
R4 aCGtcT
```

we have four reads that each clearly belong to one haplotype (reads R1 and R4 belong to one, reads R2 and R3 to the other). Real data contains read errors and hence such a bipartition is not always possible. One solution is to remove reads from the set until a solution is possible.

This can be formalized in the *Minimum Fragment Removal (MFR) problem*, which is defined as follows: Given a set of aligned sequencing reads, this set defines a conflict graph in which the nodes correspond to fragments and an edge is drawn between two nodes if the corresponding fragments have different nucleotides at a SNP position. The task is to remove the minimum number of nodes in this graph such that resulting subgraph is bipartite.

Consider the following read input:

```
R1 aCGtcT
R2 aATtcGt
R3 aGTtcGtt
R4 aCGtcT
```

(a) Draw the conflict graph for the given MFR problem.

(b) Formulate the MFR problem (objective function, constraints) for this input (hint 1: define x_r to be a binary variable which is one if the node corresponding to read r is removed from the graph. hint 2: remember that a bipartite graph has no odd cycles).

(c) Give all minimal solutions to the problem.

(d) Consider a relaxation of the MFR problem which consists of no constraints. With what strategy could you find violated contraints given your solution vector (i.e. solve the separation problem)? (hint 1: map the solution to the conflict graph and use a standard graph algorithm). It suffices to describe the mapping and the general idea.

Solution

(a) clear

(b) in general

$$\begin{array}{llll} \min & \sum_{f\in\mathcal{F}} x_f & & \\ \text{w.r.t.} & & & \\ & \sum_{f\in C} x_f & \geq & 1 & \forall C \in \mathcal{C} \\ & x_f & \in & \{0,1\} & \forall f \in \mathcal{F} \end{array}$$

for odd cycles $\mathcal{C}$.
In this case:
$x_2 + x_3 + x_4 \geq 1$
$x_1 + x_2 + x_4 \geq 1$
+ trivial constraints.

(c) $(0, 1, 0, 0)$
$(0, 0, 1, 0)$

(d) map all edges to the graph for which $x_i = 0$. Look for cycles using a graph algorithm. For each odd cycles add odd cycle constraint. If no odd cycle is found we are optimal.

4. [*12+9=21 Points*] **(Constraints)**

(a) Decide which of the following constraints can be re-formulated into LP/MILP/CP constraints in general?

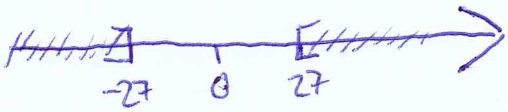
Assume x and y to be continuous variables in the case of LPs and MILPs and $x, y \in \{-50, \ldots, 50\}$ for CPs.

	Constraint	LP	MILP	CP
	$x \geq y$	X	X	X
$x \geq 3 \iff$	$x^3 \geq 27$	X	X	X
	$x \leq -3.14y$	X	X	X
$(x \geq 27) \vee (x \leq -27) \iff$	$\lvert x \rvert \geq 27$		X	X
$(x=0) \wedge (x=1) \iff$	$\lvert x \rvert = -27$	X	X X	X
$(x \leq 27) \wedge (x \geq -27) \iff$	$\lvert x \rvert \leq 27$	X		X

You get 2 points for a *correct and complete* answer, 0 points for a correct but incomplete answer, 0 points for a wrong answer.

(b) Choose three different of the above constraints. Show for one of them, how to add it to an LP, show for the next, how to add it to an MILP, and show for the third, how to add it to an CP.

$|x| \geq 27$, $x \in \mathbb{R}$ not convex $\Rightarrow$ not LP in general



$|x| \geq 27$

$\iff (x \geq 27) \vee (x \leq -27)$

$\hat{=}$ $(x \geq 27)$ or $(x \leq -27)$ in CP

$\hat{=}$ $(x + Mb \geq 27) \wedge (x - M(1-b) \leq -27)$ in MILP

$b \in \{0, 1\}$

5. [*10 Points*] **(Optimality)**

Describe the difference between global and local optimality of a discrete optimization problem on an example of your choice.