Complexity of linear programming

Theorem (Khachyian 79) The following problems are solvable in polynomial time:

- Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a vector $b \in \mathbb{Q}^m$, decide whether $Ax \leq b$ has a solution $x \in \mathbb{Q}^n$, and if so, find one.
- (Linear programming problem) Given a matrix $A \in \mathbb{Q}^{m \times n}$ and vectors $b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$, decide whether $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Q}^n\}$ is infeasible, finite, or unbounded. If it is finite, find an optimal solution. If it is unbounded, find a feasible solution x_0, and find a vector $d \in \mathbb{Q}^n$ with $Ad \leq 0$ and $c^T d > 0$.

Complexity of constraint solving: Overview

Satisfiability	over $\mathbb{Q}$	over $\mathbb{Z}$	over $\mathbb{N}$
Linear equations	polynomial	polynomial	NP-complete
Linear inequalities	polynomial	NP-complete	NP-complete

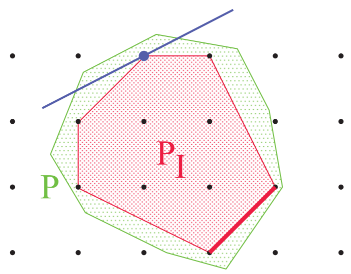
Satisfiability	over $\mathbb{R}$	over $\mathbb{Z}$
Linear constraints	polynomial	NP-complete
Non-linear constraints	decidable	undecidable

Integer Linear Optimization (ILP)

- $z_{IP} = \max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$
- $z_{LP} = \max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$ *linear (programming) relaxation*
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ *real feasible points*
- $S = \{x \in \mathbb{Z}^n \mid Ax \leq b\} = P \cap \mathbb{Z}^n$ *integer feasible points*
- **Basic properties**
  - If $P = \emptyset$, then $S = \emptyset$.
  - If z_{LP} is finite, then $S = \emptyset$ or $z_{IP} \leq z_{LP}$ is finite.
  - If $z_{LP} = \infty$, then $S = \emptyset$ or $z_{IP} = \infty$.

Integer hull

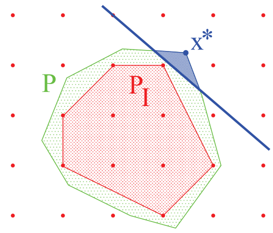
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, $S = \{x \in \mathbb{Z}^n \mid Ax \leq b\} = P \cap \mathbb{Z}^n$
- $P_I = conv(S)$ *integer hull*
- **Theorem:** P_I is again a polyhedron
- Vertices of P_I belong to S
- $\max\{c^T x \mid x \in S\} = \max\{c^T x \mid x \in conv(S)\}$



$\rightsquigarrow$ reduce integer linear optimization to linear optimization?

Cutting planes

conv(S) is very hard to compute $\rightsquigarrow$ approximation by cutting planes



- Solve the linear relaxation
$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$$
and compute a basic feasible solution x^*.
- If $x^* \in \mathbb{Z}^n$, the integer linear program has been solved.
- If $x^* \notin \mathbb{Z}^n$, generate a *cutting plane* $a^T x \leq \beta$, which is satisfied by all integer points in P, but which cuts off the fractional vertex x^* of P.
- Add the inequality $a^T x \leq \beta$ to the system $Ax \leq b$ and solve the relaxation again.

References

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