

## Duality

- *Primal problem:*  $z_P = \max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$  (P)
- *Dual problem:*  $w_D = \min\{b^T u \mid A^T u = c, u \geq 0\}$  (D)

General form

(P)	(D)
max $c^T x$	min $b^T u$
w.r.t. $A_{i*}x \leq b_i, i \in M_1$	w.r.t. $u_i \geq 0, i \in M_1$
$A_{i*}x \geq b_i, i \in M_2$	$u_i \leq 0, i \in M_2$
$A_{i*}x = b_i, i \in M_3$	$u_i$ free, $i \in M_3$
$x_j \geq 0, j \in N_1$	$(A_{*j})^T u \geq c_j, j \in N_1$
$x_j \leq 0, j \in N_2$	$(A_{*j})^T u \leq c_j, j \in N_2$
$x_j$ free, $j \in N_3$	$(A_{*j})^T u = c_j, j \in N_3$

## Duality <sup>(2)</sup>

Note: The dual of the dual is the primal.

primal	max	min	dual
constraints	$\leq b_i$	$\geq 0$	variables
	$\geq b_i$	$\leq 0$	
	$= b_i$	free	
variables	$\geq 0$	$\geq c_j$	constraints
	$\leq 0$	$\leq c_j$	
	free	$= c_j$	

## Duality theorems

- *Weak duality:* If  $x^*$  is primal feasible and  $u^*$  is dual feasible, then

$$c^T x^* \leq z_P \leq w_D \leq b^T u^*.$$

- *Strong duality*
  - If (P) and (D) both have feasible solutions, then both programs have optimal solutions and the optimum values of the objective functions are equal.
  - If one of the programs (P) or (D) has no feasible solution, then the other is either unbounded or has no feasible solution.
  - If one of the programs (P) or (D) is unbounded, then the other has no feasible solution.
- *Only four possibilities:*
  1.  $z_P$  and  $w_D$  are both finite and equal.
  2.  $z_P = +\infty$  and (D) is infeasible.
  3.  $w_D = -\infty$  and (P) is infeasible.
  4. (P) and (D) are both infeasible.

## Maximum flow and duality

- Primal problem

$$\begin{aligned} \max \quad & \sum_{e:\text{source}(e)=s} x_e - \sum_{e:\text{target}(e)=s} x_e \\ \text{s.t.} \quad & \sum_{e:\text{target}(e)=v} x_e - \sum_{e:\text{source}(e)=v} x_e = 0, \quad \forall v \in V \setminus \{s, t\} \\ & 0 \leq x_e \leq c_e, \quad \forall e \in E \end{aligned}$$

- Dual problem

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e y_e \\ \text{s.t.} \quad & z_w - z_v + y_e \geq 0, \quad \forall e = (v, w) \in E \\ & z_s = 1, z_t = 0 \\ & y_e \geq 0, \quad \forall e \in E \end{aligned}$$

## Maximum flow and duality <sup>(2)</sup>

- Let  $(y^*, z^*)$  be an optimal solution of the dual.
- Define  $S = \{v \in V \mid z_v^* > 0\}$  and  $T = V \setminus S$ .
- $(S, T)$  is a minimum cut.
- Max-flow min-cut theorem is a special case of linear programming duality.

## Farkas Lemma

**Theorem.** Suppose  $A \in \mathbb{Q}^{m \times n}$  and  $b \in \mathbb{Q}^m$ .

1. The system  $Ax \leq b$  has no solution  $x \in \mathbb{Q}^n$  if and only if there exists  $u \in \mathbb{Q}^m, u \geq 0$  such that  $u^T A = 0$  and  $u^T b = -1$ .
2. If  $Ax \leq b$  is solvable, then an inequality  $c^T x \leq \delta$  with  $c \in \mathbb{Q}^n$  and  $\delta \in \mathbb{Q}$  is satisfied by all rational solutions of  $Ax \leq b$  if and only if there exists  $u \in \mathbb{Q}^m, u \geq 0$  such that  $u^T A = c^T$  and  $u^T b \leq \delta$ .

**Rules for reasoning with linear inequalities:**

$$\text{nonneg\_lin\_com: } \frac{Ax \leq b}{(u^T A)x \leq u^T b} \quad \text{if } \begin{cases} u \in \mathbb{Q}^m, \\ u \geq 0 \end{cases}$$

$$\text{weak\_rhs: } \frac{a^T x \leq \beta}{a^T x \leq \beta'} \quad \text{if } \beta \leq \beta'$$