Linear programming

Optimization Problems

- *General optimization* problem

$$\max\{z(x) \mid f_j(x) \leq 0, x \in D\} \text{ or } \min\{z(x) \mid f_j(x) \leq 0, x \in D\}$$

  where $D \subseteq \mathbb{R}^n, f_j : D \to \mathbb{R}$, for $j = 1, \dots, m$, $z : D \to \mathbb{R}$.

- *Linear optimization* problem

$$\max\{c^T x \mid Ax \overset{\leq}{\underset{\geq}{=}} b, x \in \mathbb{R}^n\}, \text{ with } c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

- *Integer* optimization problem: $x \in \mathbb{Z}^n$
- *0-1* optimization problem: $x \in \{0,1\}^n$

Feasible and optimal solutions

- Consider the optimization problem

$$\max\{z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \dots, m\}$$

- A *feasible solution* is a vector $x^* \in D \subseteq \mathbb{R}^n$ such that $f_j(x^*) \leq 0$, for all $j = 1, \dots, m$.
- The set of all feasible solutions is called the *feasible region*.
- An *optimal solution* is a feasible solution such that

$$z(x^*) = \max\{z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \dots, m\}.$$

- Feasible or optimal solutions
  - need not exist,
  - need not be unique.

Transformations

- $\min\{z(x) \mid x \in S\} = \max\{-z(x) \mid x \in S\}$.
- $f(x) \geq a$ if and only if $-f(x) \leq -a$.
- $f(x) = a$ if and only if $f(x) \leq a \wedge -f(x) \leq -a$.

Lemma

Any linear programming problem can be brought to the form

$$\max\{c^T x \mid Ax \leq b\} \text{ or } \max\{c^T x \mid Ax = b, x \geq 0\}.$$

Proof: a) $a^T x \leq \beta \rightsquigarrow a^T x + x' = \beta, x' \geq 0$ *(slack variable)*
b) x free $\rightsquigarrow x = x^+ - x^-, \; x^+, x^- \geq 0$.

Practical problem solving

1. Model building
2. Model solving
3. Model analysis

Example: Production problem

A firm produces n different goods using m different raw materials.

- b_i: available amount of the i-th raw material
- a_{ij}: number of units of the i-th material needed to produce one unit of the j-th good
- c_j: revenue for one unit of the j-th good.

Decide how much of each good to produce in order to maximize the total revenue $\rightsquigarrow$ *decision variables* x_j.

Linear programming formulation

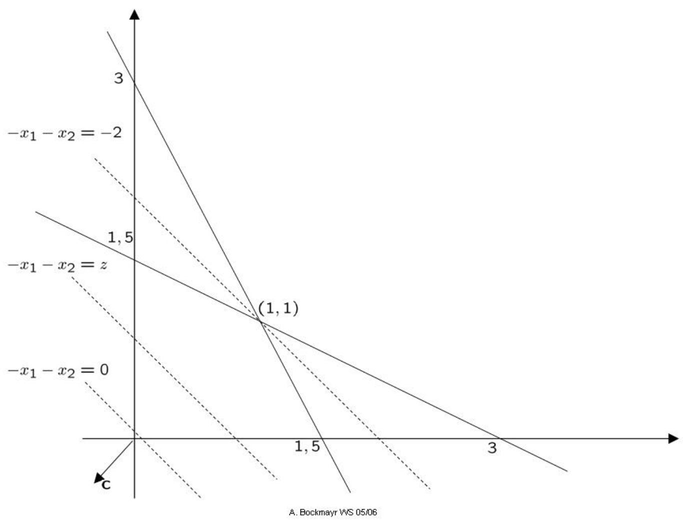
$$
\begin{array}{lccccccc}
\max & c_1 x_1 & + & \cdots & + & c_n x_n & & \\
\text{w.r.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \leq & b_1, \\
& \vdots & & & & \vdots & & \\
& a_{m1} x_1 & + & \cdots & + & a_{mn} x_n & \leq & b_m, \\
& x_1, & & \ldots & & , x_n & \geq & 0.
\end{array}
$$

In matrix notation:

$$\max\{c^T x \mid Ax \leq b, x \geq 0\},$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, x \in \mathbb{R}^n$.

Geometric illustration



$$
\begin{array}{lccccc}
\max & x_1 & + & x_2 & & \\
\text{w.r.t.} & x_1 & + & 2x_2 & \leq & 3 \\
& 2x_1 & + & x_2 & \leq & 3 \\
& x_1 & , & x_2 & \geq & 0
\end{array}
$$