Complexity of linear programming

Theorem (Khachyian 79) The following problems are solvable in polynomial time:

- Given a matrix $A \in \mathbb{Q}^{m\times n}$ and a vector $b \in \mathbb{Q}^m$, decide whether $Ax \leq b$ has a solution $x \in \mathbb{Q}^n$, and if so, find one.
- (Linear programming problem) Given a matrix $A \in \mathbb{Q}^{m\times n}$ and vectors $b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$, decide whether $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Q}^n\}$ is infeasible, finite, or unbounded. If it is finite, find an optimal solution. If it is unbounded, find a feasible solution x_0, and find a vector $d \in \mathbb{Q}^n$ with $Ad \leq 0$ and $c^T d > 0$.

Ellipsoids

- A set $E \subseteq \mathbb{R}^n$ is called an *ellipsoid* if there exists a vector $c \in \mathbb{R}^n$, called the *center* of E, and a positive definite matrix $D \in \mathbb{R}^{n\times n}$ such that

$$E = E(c, D) = \{x \in \mathbb{R}^n \mid (x-c)^T D^{-1}(x-c) \leq 1\}.$$

- A symmetric matrix D is *positive definite*, if $x^T Dx > 0$, for all $x \in \mathbb{R}^n \setminus \{0\}$. Any positive definite matrix is non-singular, and D^{-1} is again positive definite.
- The unit ball $B(0,1) = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ around 0 in the Euclidean norm is identical with the ellipsoid $E(0, I)$.
- For every positive definite matrix D there exists a unique positive definite matrix $D^{1/2}$ such that $D = D^{1/2}D^{1/2}$.
- It follows that $E(c, D) = D^{1/2}B(0,1) + c \rightsquigarrow$ *every ellipsoid is the image of the unit ball under a bijective affine transformation.*

Theorem

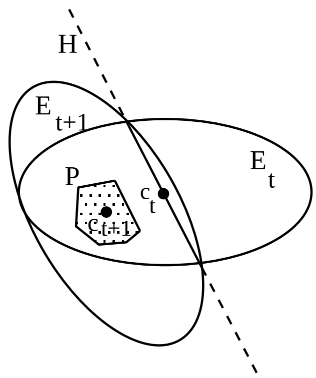
Let $E_t = E(c_t, D_t) \subseteq \mathbb{R}^n$ be an ellipsoid and let $a \in \mathbb{R}^n$ be a non-zero vector.

Consider the halfspace $H = \{x \in \mathbb{R}^n \mid a^T x \leq a^T c_t\}$ defined by the hyperplane in direction a through c_t.

Let $c_{t+1} = c_t - \frac{1}{n+1} d_t$ and $D_{t+1} = \frac{n^2}{n^2-1}(D_t - \frac{2}{n+1} d_t d_t{}^T)$, where $d_t = \frac{1}{\sqrt{a^T D_t a}} D_t a$.

Then $E_{t+1} = E(c_{t+1}, D_{t+1})$ is an ellipsoid such that

- $E_t \cap H \subset E_{t+1}$
- $vol(E_{t+1}) < e^{-\frac{1}{2n}} vol(E_t)$



Ellipsoid method

- Consider the polyhedron $P = \{x \in \mathbb{F}^n \mid Ax \leq b\}$, $\quad A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$, and assume that P is either empty or bounded and full-dimensional.
- Construct a sequence of ellipsoids $E_t, t \in \mathbb{N}$, such that all E_t contain P and such that $vol(E_t)$ gets smaller and smaller.

- Suppose we have computed the ellipsoid $E_t = E(c_t, D_t)$.
  - If $c_t \in P$, then P is non-empty and the algorithm terminates.
  - If $c_t \notin P$, there exists an inequality $a^T x \leq \beta$ in the system $Ax \leq b$ such that $a^T c_t > \beta$.
- It follows that P is contained in the intersection $E_t \cap H$ of E_t with the half-space $H = \{x \in \mathbb{R}^n \mid a^T x \leq a^T c_t\}$.
- Now we can construct a new ellipsoid E_{t+1} containing the half-ellipsoid $H \cap E_t$ such that the volume of E_{t+1} is only a fraction of the volume of E_t.

Overview of constraint solving problems

Satisfiability	over $\mathbb{Q}$	over $\mathbb{Z}$	over $\mathbb{N}$
Linear equations	polynomial	polynomial	NP-complete
Linear inequalities	polynomial	NP-complete	NP-complete

Satisfiability	over $\mathbb{R}$	over $\mathbb{Z}$
Linear constraints	polynomial	NP-complete
Non-linear constraints	decidable	undecidable

Duality

- *Primal problem:* $z_P = \max\{\mathbf{c}^\mathbf{T} x \mid Ax \leq b, \ x \in \mathbb{R}^n\}$ (P)
- *Dual problem:* $w_D = \min\{b^T u \mid A^T u = \mathbf{c}, \ u \geq 0\}$ (D)

General form

(P)			(D)		
min	$c^T x$		max	$u^T b$	
w.r.t.	$A_{i*} x \geq b_i$,	$i \in M_1$	w.r.t	$u_i \geq 0$,	$i \in M_1$
	$A_{i*} x \leq b_i$,	$i \in M_2$		$u_i \leq 0$,	$i \in M_2$
	$A_{i*} x = b_i$,	$i \in M_3$		u_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$		$(A_{*j})^T u \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$		$(A_{*j})^T u \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$		$(A_{*j})^T u = c_j$,	$j \in N_3$

Duality theorems

- *Weak duality:* If x^* is primal feasible and u^* is dual feasible, then
$$c^T x^* \leq z_P \leq w_D \leq b^T u^*.$$
- *Strong duality*
  - If (P) and (D) both have feasible solutions, then both programs have optimal solutions and the optimum values of the objective functions are equal.
  - If one of the programs (P) or (D) has no feasible solution, then the other is either unbounded or has no feasible solution.
  - If one of the programs (P) or (D) is unbounded, then the other has no feasible solution.

- *Only four possibilities:*

    1. z_P and w_D are both finite and equal.
    2. $z_P = +\infty$ and (D) is infeasible.
    3. $w_D = -\infty$ and (P) is infeasible.
    4. (P) and (D) are both infeasible.

Maximum flow and duality

- Primal problem

$$
\begin{aligned}
\max \quad & \sum_{e:\text{source}(e)=s} x_e - \sum_{e:\text{target}(e)=s} x_e \\
\text{s.t.} \quad & \sum_{e:\text{target}(e)=v} x_e - \sum_{e:\text{source}(e)=v} x_e = 0, \quad \forall v \in V \setminus \{s,t\} \\
& 0 \leq x_e \leq c_e, \qquad \forall e \in E
\end{aligned}
$$

- Dual problem

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e \\
\text{s.t.} \quad & z_w - z_v + y_e \geq 0, \quad \forall e = (v,w) \in E \\
& z_s = 1, z_t = 0 \\
& y_e \geq 0, \qquad \forall e \in E
\end{aligned}
$$

Maximum flow and duality (2)

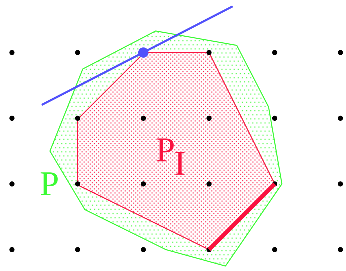
- Let (y^*, z^*) be an optimal solution of the dual.
- Define $S = \{v \in V \mid z_v^* > 0\}$ and $T = V \setminus S$.
- (S,T) is a minimum cut.
- Max-flow min-cut theorem is a special case of linear programming duality.

Integer Linear Optimization (ILP)

- $z_{IP} = \max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- $z_{LP} = \max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$ *linear (programming) relaxation*
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ *real feasible points*
- $S = \{x \in \mathbb{Z}^n \mid Ax \leq b\} = P \cap \mathbb{Z}^n$ *integer feasible points*
- **Basic properties**
    - If $P = \emptyset$, then $S = \emptyset$.
    - If z_{LP} is finite, then $S = \emptyset$ or $z_{IP} \leq z_{LP}$ is finite.
    - If $z_{LP} = \infty$, then $S = \emptyset$ or $z_{IP} = \infty$.

Integer hull

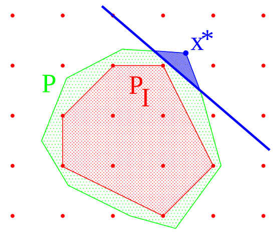
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, \;\; S = \{x \in \mathbb{Z}^n \mid Ax \leq b\} = P \cap \mathbb{Z}^n$
- $P_I = conv(S)$ *integer hull*
- **Theorem:** P_I is again a polyhedron
- Vertices of P_I belong to S
- $\max\{c^T x \mid x \in S\} = \max\{c^T x \mid x \in conv(S)\}$



$\rightsquigarrow$ reduce integer linear optimization to linear optimization?

Cutting planes

$conv(S)$ is very hard to compute $\rightsquigarrow$ approximation by cutting planes



- Solve the linear relaxation
$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$$
and compute a basic feasible solution x^*.
- If $x^* \in \mathbb{Z}^n$, the integer linear program has been solved.
- If $x^* \notin \mathbb{Z}^n$, generate a *cutting plane* $a^T x \leq \beta$, which is satisfied by all integer points in P, but which cuts off the fractional vertex x^* of P.
- Add the inequality $a^T x \leq \beta$ to the system $Ax \leq b$ and solve the relaxation again.

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