

## Integer vs. constraint programming

### Practical Problem Solving

- Model building: Language
- Model solving: Algorithms

### IP vs. CP: Language

	IP	CP
Variables	(mostly) 0-1	Finite domain
Constraints	Linear equations and inequalities	Arithmetic constraints <b>Symbolic/global constraints</b>

### Example

- Variables:  $x_1, \dots, x_n \in \{0, \dots, m-1\}$
- Constraint: Pairwise different values

### Example <sup>(2)</sup>

- Integer programming: Only linear equations and inequalities

$$\begin{aligned}
 x_i \neq x_j &\iff x_i < x_j \vee x_i > x_j \\
 &\iff x_i \leq x_j - 1 \vee x_i \geq x_j + 1
 \end{aligned}$$

- Eliminating disjunction

$$\begin{aligned}
 x_i - x_j + 1 \leq my_1, \quad x_j - x_i + 1 \leq my_2, \quad y_1 + y_2 = 1, \\
 y_1, y_2 \in \{0, 1\}, \quad 0 \leq x_i, x_j \leq m - 1,
 \end{aligned}$$

- New variables:  $z_{ik} = 1$  iff  $x_i = k$ ,  $i = 1, \dots, n$ ,  $k = 0, \dots, m - 1$

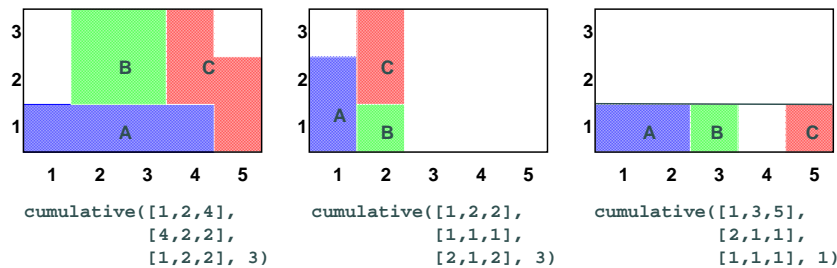
$$z_{i0} + \dots + z_{im-1} = 1, \quad z_{1k} + \dots + z_{nk} \leq 1,$$

- Constraint programming  $\rightsquigarrow$  **symbolic constraint**

$$\text{alldifferent}(x_1, \dots, x_n)$$

### Symbolic/global constraints

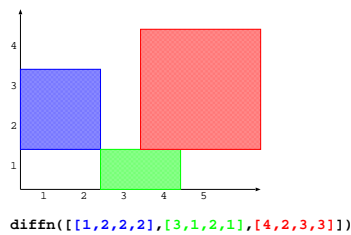
- `alldifferent` ( $[x_1, \dots, x_n]$ )
- `cumulative` ( $[s_1, \dots, s_n], [d_1, \dots, d_n], [r_1, \dots, r_n], c, e$ ).
  - $n$  tasks: starting time  $s_i$ , duration  $d_i$ , resource demand  $r_i$
  - resource capacity  $c$ , completion time  $e$



### Diffn Constraint

Beldiceanu/Contejean'94

- Nonoverlapping of  $n$ -dimensional rectangles  $[O_1, \dots, O_n, L_1, \dots, L_n]$ , where  $O_i$  (resp.  $L_i$ ) denotes the origin (resp. length) in dimension  $i$
- `diffn` ( $[[O_{11}, \dots, O_{1n}, L_{11}, \dots, L_{1n}], \dots, [O_{m1}, \dots, O_{mn}, L_{m1}, \dots, L_{mn}]]$ )



- General form: `diffn` (Rectangles, Min\_Vol, Max\_Vol, End, Distances, Regions)

### IP vs. CP: Algorithms

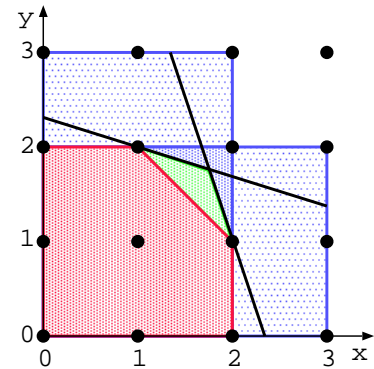
	IP	CP
<i>Inference</i>	Linear programming Cutting planes	Domain filtering Constraint propagation
<i>Search</i>	Branch-and-relax Branch-and-cut	Branch-and-bound
Bounds on the objective function	Two-sided	One-sided

## Local vs. global reasoning

Linear arithmetic constraints

$$\begin{aligned} 3x + y &\leq 7, \\ 3y + x &\leq 7, \\ x + y &= z, \\ x, y &\in \{0, \dots, 3\} \end{aligned}$$

$$\begin{aligned} \text{CP} \quad &x, y \leq 2, z \leq 4 \\ \text{LP} \quad &x, y \leq 2, z \leq 3.5 \\ \text{IP} \quad &x, y \leq 2, z \leq 3 \end{aligned}$$



Global reasoning in CP ?  $\rightsquigarrow$  global constraints!

## Global reasoning in CP

Example

- $x_1, x_2, x_3 \in \{0, 1\}$
- pairwise different values
- **Local** consistency. 3 disequalities:  $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$   
 $\rightsquigarrow x_1, x_2, x_3 \in \{0, 1\}$ , i.e., no domain reduction is possible
- **Global** constraint:  $\text{alldifferent}(x_1, x_2, x_3)$   
 $\rightsquigarrow$  detects infeasibility (uses bipartite matching)

Global reasoning in CP: inside global constraints

## Summary

	ILP	CP(FD)
<i>Language</i>	Linear arithmetic —	Arithmetic constraints <b>Symbolic constraints</b>
<i>Algorithms</i>	Global consistency (LP) Cutting planes	Local consistency Domain reduction
	Branch-and-bound Branch-and-cut	User-defined enumeration

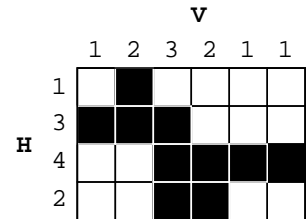
- Symbolic constraints  $\rightsquigarrow$  more expressivity + more efficiency
- Unifying framework for CP and IP: *Branch-and-infer* (Bockmayr/Kasper 98), . . . , SCIP

## Discrete Tomography

- Binary matrix with  $m$  rows and  $n$  columns
  - Horizontal projection numbers  $(h_1, \dots, h_m)$
  - Vertical projection numbers  $(v_1, \dots, v_n)$

- *Properties*

- Horizontal convexity (h)
- Vertical convexity (v)
- Connectivity (polyomino) (p)



- *Complexity* (Woeginger'01)
  - polynomial:  $(p, v, h)$
  - NP-complete:  $(p, v), (p, h), (v, h), (v), (h), (p)$

### IP Model

- *Variables*  $x_{ij} = \begin{cases} 0 & \text{cell}(i, j) \text{ is labeled white} \\ 1 & \text{cell}(i, j) \text{ is labeled black} \end{cases}$

- *Constraints I: Projections*

$$\sum_{j=1}^n x_{ij} = h_i, \quad \sum_{i=1}^m x_{ij} = v_j$$

- *Constraints II: Convexity*



$$h_i x_{ik} + \sum_{l=k+h_i}^n x_{il} \leq h_i, \quad v_j x_{kj} + \sum_{l=k+v_j}^m x_{lj} \leq v_j,$$

### IP Model (contd)

- *Constraints III: Connectivity*

$$\sum_{k=j}^{j+h_i-1} x_{ik} - \sum_{k=j}^{j+h_i-1} x_{i+1,k} \leq h_i - 1$$

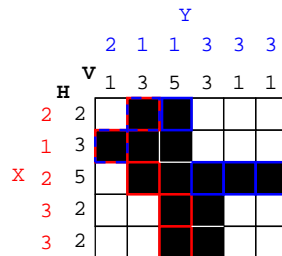


- Various linear arithmetic models possible, e.g. convexity
- Enormous differences in size and running time, e.g. 1 day vs. < 1 sec
- Large number of constraints ( $\sim 3mn$  in the above model)

### Finite Domain Model

- Variables

- $x_i$  start of horizontal convex block in row  $i$ , for  $1 \leq i \leq m$
- $y_j$  start of vertical convex block in column  $j$ , for  $1 \leq j \leq n$



- Domain

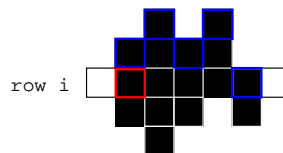
- $x_i \in [1, \dots, n - h_i + 1]$ , for  $1 \leq i \leq m$
- $y_j \in [1, \dots, m - v_j + 1]$ , for  $1 \leq j \leq n$

### Conditional Propagation

- Projection/Convexity modelled by FD variables
- Compatibility of  $x_i$  and  $y_j$

$$x_i \leq j < x_i + h_i \iff y_j \leq i < y_j + v_j$$

for  $1 \leq i \leq m$  and  $1 \leq j \leq n$

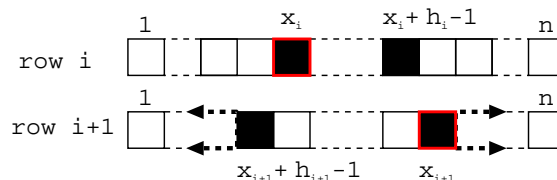


- Conditional propagation

$$\text{if } x_i \leq j \text{ then (if } j < x_i + h_i \text{ then } (y_j \leq i, i < y_j + v_j))$$

### Finite Domain Model (contd)

- Connectivity



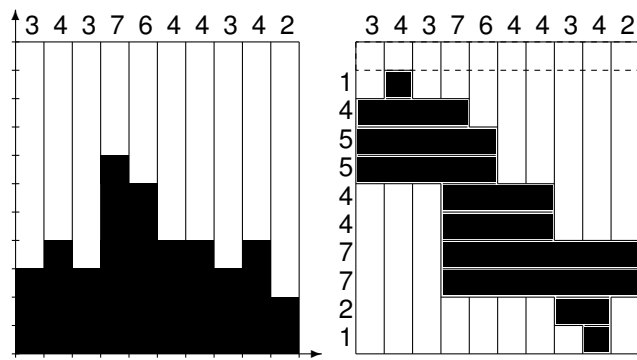
- Block  $i$  must start before the end of block  $i + 1$

$$x_i \leq x_{i+1} + h_{i+1} - 1, \text{ for } 1 \leq i \leq m - 1$$

- Block  $i + 1$  must start before the end of block  $i$

$$x_{i+1} \leq x_i + h_i - 1, \text{ for } 1 \leq i \leq m - 1$$

### Cumulative



### 2d and 3d Diffn Model

