Integer vs. constraint programming

Practical Problem Solving

- Model building: Language
- Model solving: Algorithms

IP vs. CP: Language

	IP	CP
Variables	(mostly) 0-1	Finite domain
Constraints	Linear equations and inequalities	Arithmetic constraints **Symbolic/global constraints**

Example

- Variables: $x_1, \dots, x_n \in \{0, \dots, m-1\}$
- Constraint: Pairwise different values

Example (2)

- Integer programming: Only linear equations and inequalities

$$\begin{array}{lllllllll} x_i \neq x_j & \Longleftrightarrow & x_i & < & x_j & \vee & x_i & > & x_j \\ & \Longleftrightarrow & x_i & \leq & x_j - 1 & \vee & x_i & \geq & x_j + 1 \end{array}$$

- Eliminating disjunction

$$x_i - x_j + 1 \leq my_1, \quad x_j - x_i + 1 \leq my_2, \quad y_1 + y_2 = 1,$$
$$y_1, y_2 \in \{0, 1\}, \quad 0 \leq x_i, x_j \leq m - 1,$$

- New variables: $z_{ik} = 1$ *iff* $x_i = k$, $i = 1, \dots, n$, $k = 0, \dots, m-1$

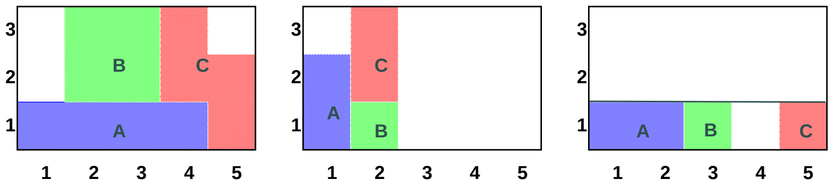
$$z_{i0} + \dots + z_{im-1} = 1, \quad z_{1k} + \dots + z_{nk} \leq 1,$$

- Constraint programming $\rightsquigarrow$ **symbolic constraint**

$$\texttt{alldifferent}(x_1, \dots, x_n)$$

Symbolic/global constraints

- `alldifferent`$([x_1, \ldots, x_n])$
- `cumulative`$([s_1, \ldots, s_n], [d_1, \ldots, d_n], [r_1, \ldots, r_n], c, e)$.
  - n tasks: starting time s_i, duration d_i, resource demand r_i
  - resource capacity c, completion time e

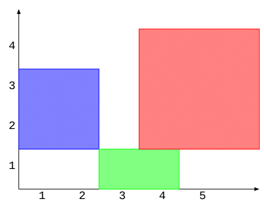



```
cumulative([1,2,4],        cumulative([1,2,2],        cumulative([1,3,5],
           [4,2,2],                   [1,1,1],                   [2,1,1],
           [1,2,2], 3)                [2,1,2], 3)                [1,1,1], 1)
```

Diffn Constraint

Beldiceanu/Contejean'94

- Nonoverlapping of n-dimensional rectangles $[O_1, \ldots, O_n, L_1, \ldots, L_n]$, where O_i (resp. L_i) denotes the origin (resp. length) in dimension i
- `diffn`$([[O_{11}, \ldots, O_{1n}, L_{11}, \ldots, L_{1n}], \ldots, [O_{m1}, \ldots, O_{mn}, L_{m1}, \ldots, L_{mn}]])$



`diffn([[1,2,2,2],[3,1,2,1],[4,2,3,3]])`

- General form: `diffn(Rectangles,Min_Vol,Max_Vol,End,Distances,Regions)`

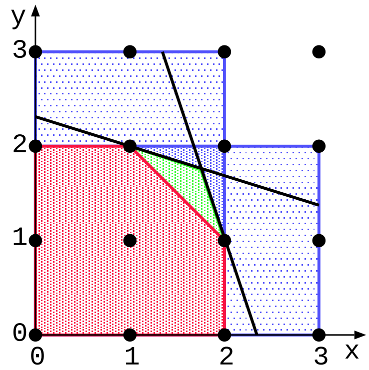
IP vs. CP: Algorithms

	IP	CP
Inference	Linear programming Cutting planes	Domain filtering Constraint propagation
Search	Branch-and-relax Branch-and-cut	Branch-and-bound
Bounds on the objective function	Two-sided	One-sided

Local vs. global reasoning

Linear arithmetic constraints

$$\begin{array}{rcl} 3x+y & \leq & 7, \\ 3y+x & \leq & 7, \\ x+y & = & z, \\ x,y & \in & \{0,\dots,3\} \end{array}$$



CP	$x,y \leq 2$, $z \leq 4$
LP	$x,y \leq 2$, $z \leq 3.5$
IP	$x,y \leq 2$, $z \leq 3$

Global reasoning in CP ? $\rightsquigarrow$ global constraints!

Global reasoning in CP

Example

- $x_1, x_2, x_3 \in \{0,1\}$
- pairwise different values
- **Local** consistency. 3 disequalities: $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$
  $\rightsquigarrow$ $x_1, x_2, x_3 \in \{0,1\}$, i.e., no domain reduction is possible
- **Global** constraint: `alldifferent`(x_1, x_2, x_3)
  $\rightsquigarrow$ detects infeasibility (uses bipartite matching)

Global reasoning in CP: inside global constraints

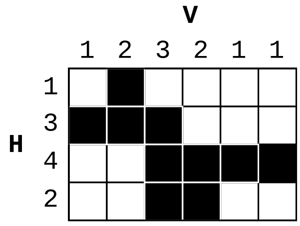
Summary

	ILP	CP(FD)
Language	Linear arithmetic —	Arithmetic constraints **Symbolic constraints**
Algorithms	Global consistency (LP) Cutting planes	Local consistency Domain reduction
	Branch-and-bound Branch-and-cut	User-defined enumeration

- Symbolic constraints $\rightsquigarrow$ more expressivity + more efficiency
- Unifying framework for CP and IP: *Branch-and-infer*
  (Bockmayr/Kasper 98), ..., SCIP

Discrete Tomography

- Binary matrix with m rows and n columns
  - Horizontal projection numbers $(h_1, \dots, h_m)$
  - Vertical projection numbers $(v_1, \dots, v_n)$
- *Properties*
  - Horizontal convexity (h)
  - Vertical convexity (v)
  - Connectivity (polyomino) (p)



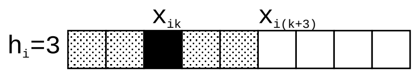
- *Complexity* (Woeginger'01)
  - polynomial: ( ), (p,v,h)
  - NP-complete: (p,v), (p,h), (v,h), (v), (h), (p)

IP Model

- *Variables* $x_{ij} = \begin{cases} 0 & \text{cell}(i,j) \text{ is labeled white} \\ 1 & \text{cell}(i,j) \text{ is labeled black} \end{cases}$
- *Constraints I:* Projections

$$\sum_{j=1}^{n} x_{ij} = h_i, \qquad \sum_{i=1}^{m} x_{ij} = v_j$$

- *Constraints II:* Convexity

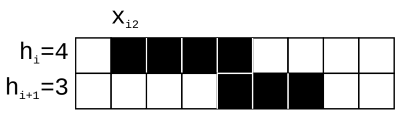


$$h_i x_{ik} + \sum_{l=k+h_i}^{n} x_{il} \leq h_i, \qquad v_j x_{kj} + \sum_{l=k+v_j}^{m} x_{lj} \leq v_j,$$

IP Model (contd)

- *Constraints III:* Connectivity

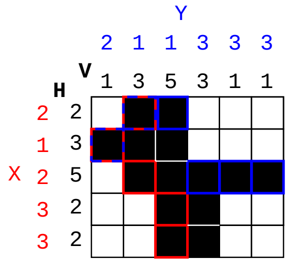
$$\sum_{k=j}^{j+h_i-1} x_{ik} - \sum_{k=j}^{j+h_i-1} x_{i+1,k} \leq h_i - 1$$



- Various linear arithmetic models possible, e.g. convexity
- Enormous differences in size and running time, e.g. 1 day vs. < 1 sec
- Large number of constraints ($\sim 3mn$ in the above model)

Finite Domain Model

- *Variables*
  - x_i start of horizontal convex block in row i, for $1 \leq i \leq m$
  - y_j start of vertical convex block in column j, for $1 \leq j \leq n$



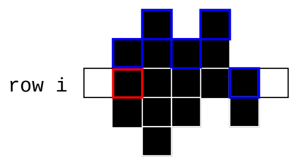
- *Domain*
  - $x_i \in [1, \ldots, n - h_i + 1]$, for $1 \leq i \leq m$
  - $y_j \in \left[1, \ldots, m - v_j + 1\right]$, for $1 \leq j \leq n$

Conditional Propagation

- Projection/Convexity modelled by FD variables
- *Compatibility* of x_i and y_j

$$x_i \leq j < x_i + h_i \iff y_j \leq i < y_j + v_j$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$

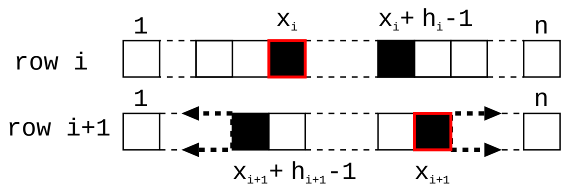


- *Conditional propagation*

$$\texttt{if } x_i \leq j \texttt{ then } (\texttt{if } j < x_i + h_i \texttt{ then } (y_j \leq i,\ i < y_j + v_j))$$

Finite Domain Model (contd)

- *Connectivity*



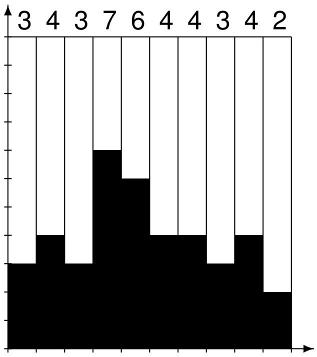
- Block i must start before the end of block $i+1$

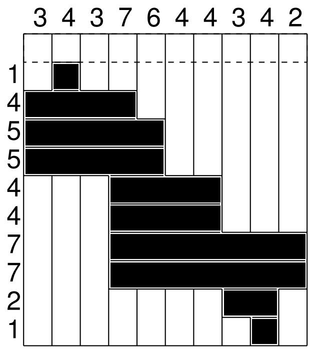
$$x_i \leq x_{i+1} + h_{i+1} - 1, \text{ for } 1 \leq i \leq m - 1$$

- Block $i+1$ must start before the end of block i

$$x_{i+1} \leq x_i + h_i - 1, \text{ for } 1 \leq i \leq m - 1$$

Cumulative





2d and 3d Diffn Model