10.1 Burrows-Wheeler transform

This exposition has been developed by David Weese. It is based on the following sources, which are all recommended reading:

1. M. Burrows and D. J. Wheeler (1994) *A Block-sorting Lossless Data Compression Algorithm*, SRC Research Report 124
2. G. Navarro and V. Mäkinen (2007) *Compressed Full-Text Indexes*, ACM Computing Surveys 39(1), Article no. 2 (61 pages), Section 5.3
3. D. Adjeroh, T. Bell, and A. Mukherjee (2008) *The Burrows-Wheeler Transform: Data Compression, Suffix Arrays and Pattern Matching*, Springer

10.2 Motivation

We have seen the suffix array to be an efficient data structure for exact string matching in a fixed text. However, for large texts like the human genome of about 3 billion basepairs, the text and the suffix array alone would consume **15 Gb** of memory. To solve the exact string matching in optimal $O(m + p)$ time (m=pattern length, p=number of occurrences) we would need an enhanced suffix array of $3 \cdot (1 + 4 + 4 + 4)$ Gb = **39 Gb** of memory.

Both cases exceed the amount of physical memory of a typical desktop computer, therefore we need a different data structure with a smaller memory footprint. Burrows and Wheeler proposed in 1994 a lossless compression algorithm (now used in bzip2). The algorithm transforms a given text into the so called *Burrows-Wheeler transform* (BWT), a permutation of the text that can be back transformed. The transformed text can in general be better compressed than the original text as in the BWT equal characters tend to form consecutive runs which can be compressed using run-length encoders.

We will see that it is possible to conduct exact searches using only the compressed BWT and some auxiliary tables.

10.3 Definitions

We consider a string T of length n. For $i, j \in \mathbb{N}$ we define:

- $[i..j] := \{i, i+1, \ldots, j\}$
- $[i..j) := [i..j-1]$
- $T[i]$ is the i-th character of T.
- $T[i..j] := T[i]T[i+1] \ldots T[j]$ is the substring from the i-th to the j-th character
- We start counting from **1**, i. e. $T = T[1..n]$
- $|T|$ denotes the string length, i. e. $|T| = n$
- The concatenation of strings X, Y is denoted as $X \cdot Y$, e. g. $T = T[1..i] \cdot T[i+1..n]$ for $i \in [1..n)$

Definition 1 (cyclic shift)**.** Let $T = T[1..n]$ be a text over the alphabet Σ that ends with unique character $T[n] = \$$, which is the lexicographically smallest character in Σ. The i-th *cyclic shift* of T is $T[i..n] \cdot T[1..i-1]$ for $i \in [1..n]$ and denoted as $T^{(i)}$.

Example 2.

$$\begin{aligned}
T &= \texttt{mississippi\$} \\
T^{(1)} &= \texttt{mississippi\$} \\
&\vdots \\
T^{(3)} &= \texttt{ssissippi\$mi} \\
&\vdots \\
T^{(n)} &= \texttt{\$mississippi}
\end{aligned}$$

10.4 Burrows-Wheeler transform

The Burrows-Wheeler transform (BWT) can be obtained by the following steps:

1. Form a conceptual matrix $\mathcal{M}$ whose rows are the n cyclic shifts of the text T
2. Lexicographically sort the rows of $\mathcal{M}$.
3. Construct the transformed text T^{bwt} by taking the last column of $\mathcal{M}$.

The transformed text T^{bwt} in the last column is also denoted as **L** (last). Notice that every row and every column of $\mathcal{M}$, hence also the transformed text L is a permutation of T. In particular the first column of $\mathcal{M}$, call it **F** (first), is obtained by lexicographically sorting the characters of T (or, equally, the characters of L).

Example 3. Form $\mathcal{M}$ and sort rows lexicographically:

		F		*L*
`mississippi$`		`$`	`mississipp`	*i*
`ississippi$m`		`i`	`$mississip`	*p*
`ssissippi$mi`		`i`	`ppi$missis`	*s*
`sissippi$mis`		`i`	`ssippi$mis`	*s*
`issippi$miss`		`i`	`ssissippi$`	*m*
`ssippi$missi`	sort	`m`	`ississippi`	*$*
`sippi$missis`	⇒	`p`	`i$mississi`	*p*
`ippi$mississ`		`p`	`pi$mississ`	*i*
`ppi$mississi`		`s`	`ippi$missi`	*s*
`pi$mississip`		`s`	`issippi$mi`	*s*
`i$mississipp`		`s`	`sippi$miss`	*i*
`$mississippi`		`s`	`sissippi$m`	*i*

The transformed string L usually contains long runs of identical symbols and therefore can be efficiently compressed using move-to-front coding, in combination with statistical coders.

10.5 Constructing the BWT

Note that when we sort the rows of $\mathcal{M}$ we are essentially sorting the suffixes of T. Hence, there is a strong relation between the matrix $\mathcal{M}$ and the suffix array A of T.

i	$T^{(i)}$		i	$T^{(i)}$		A	$T[A[j]..n]$
1	`mississippi$`		12	`$mississippi`		12	`$`
2	`ississippi$m`		11	`i$mississipp`		11	`i$`
3	`ssissippi$mi`		8	`ippi$mississ`		8	`ippi$`
4	`sissippi$mis`		5	`issippi$miss`		5	`issippi$`
5	`issippi$miss`		2	`ississippi$m`		2	`ississippi$`
6	`ssippi$missi`	sort	1	`mississippi$`	$\widehat{=}$	1	`mississippi$`
7	`sippi$missis`	⇒	10	`pi$mississip`		10	`pi$`
8	`ippi$mississ`		9	`ppi$mississi`		9	`ppi$`
9	`ppi$mississi`		7	`sippi$missis`		7	`sippi$`
10	`pi$mississip`		4	`sissippi$mis`		4	`sissippi$`
11	`i$mississipp`		6	`ssippi$missi`		6	`ssippi$`
12	`$mississippi`		3	`ssissippi$mi`		3	`ssissippi$`

Lemma 4. *The Burrows-Wheeler transform T^{bwt} can be constructed from the suffix array A of T. It holds:*

$$T^{bwt}[i] = \begin{cases} T[A[i]-1] & \text{if } A[i] > 1 \\ \$ & \text{else} \end{cases}$$

Proof: Since T is terminated with the special character \$, which is lexicographically smaller than any other character and occurs only at the end of T, a comparison of two shifts ends at latest after comparing a \$. Hence the characters right of the \$ do not influence the order of the cyclic shifts and they are sorted exactly like the suffices of T. For each suffix starting at position $A[i]$ the last column contains the preceding character at position $A[i]-1$ (or n resp.).

Corollary 5. *The Burrows-Wheeler transform of a text of length n can be constructed in $O(n)$ time.*

Corollary 6. *The i-th row of $\mathcal{M}$ contains the $A[i]$-th cyclic shift of T, i. e. $\mathcal{M}_i = T^{(A[i])}$.*

10.6 Reverse transform

One interesting property of the Burrows-Wheeler transform T^{bwt} is that the original text T can be reconstructed by a reverse transform without any extra information. Therefore we need the following definition:

Definition 7 (L-to-F mapping)**.** Let $\mathcal{M}$ be the sorted matrix of cyclic shifts of the text T. LF is a function $\mathsf{LF} : [1..n] \rightarrow [1..n]$ that maps the rank of a cyclic shift X to the rank of $X^{(n)}$ which is X shifted by one to the right:

$$\mathsf{LF}(l) = f \Leftrightarrow \mathcal{M}_f = \mathcal{M}_l^{(n)}$$

LF represents a one-to-one correspondence between elements of F and elements of L, and $L[i] = F[\mathsf{LF}[i]]$ for all $i \in [1..n]$. Corresponding characters stem from the same position in the text. That can be concluded from the following equivalence:

$$\begin{aligned} \mathsf{LF}(l) = f \quad &\Leftrightarrow \quad \mathcal{M}_f = \mathcal{M}_l{}^{(n)} \\ &\Leftrightarrow \quad T^{(A[f])} = T^{(A[l])^{(n)}} \\ &\Leftrightarrow \quad T^{(A[f])} = T^{(A[l]+n-1)} \\ &\Leftrightarrow \quad A[f] \equiv A[l] + (n-1) \qquad (\text{mod n}) \end{aligned}$$

Example 8. $T =$ `mississippi$`. It holds $\mathsf{LF}(1) = 2$ as the cyclic shift in row 1 of $\mathcal{M}$ shifted by one to the right occurs in row 2. $\mathsf{LF}(2) = 7$ as the cyclic shift in row 2 of $\mathcal{M}$ shifted by one to the right occurs in row 7. For the same reason holds $\mathsf{LF}(7) = 8$.

	F		L
$T^{(12)}$	`$`	`mississipp`	$\boxed{\texttt{i}}$
$T^{(11)}$	$\boxed{\texttt{i}}$	`$mississip`	$\boxed{\texttt{p}}$
	`i`	`ppi$missis`	`s`
	`i`	`ssippi$mis`	`s`
	`i`	`ssissippi$`	`m`
	`m`	`ississippi`	`$`
$T^{(10)}$	$\boxed{\texttt{p}}$	`i$mississi`	$\boxed{\texttt{p}}$
$T^{(9)}$	$\boxed{\texttt{p}}$	`pi$mississ`	`i`
	`s`	`ippi$missi`	`s`
	`s`	`issippi$mi`	`s`
	`s`	`sippi$miss`	`i`
	`s`	`sissippi$m`	`i`

i	$\mathsf{LF}(i)$
1	**2**
2	**7**
3	9
4	10
5	6
6	1
7	**8**
8	3
9	11
10	12
11	4
12	5

Thus the first character in row f stems from position $A[f]$ in the text. That is the same position the last character in row l stems from. One important observation is that the relative order of two cyclic shifts that end with the same character is preserved after shifting them one to the right.

Observation 9 (rank preservation)**.** Let $i, j \in [1..n]$ with $L[i] = L[j]$. If $i < j$ then $\mathsf{LF}[i] < \mathsf{LF}[j]$ follows.

Proof: From $L[i] = L[j]$ and $i < j$ follows $\mathcal{M}_i[1..n-1] <_{\text{lex}} \mathcal{M}_j[1..n-1]$. Thus holds:

$$\begin{aligned} & L[i] \cdot \mathcal{M}_i[1..n-1] &<_{\text{lex}}\ & L[j] \cdot \mathcal{M}_j[1..n-1] \\ \Leftrightarrow\ & \mathcal{M}_i^{(n)} &<_{\text{lex}}\ & \mathcal{M}_j^{(n)} \\ \Leftrightarrow\ & \mathcal{M}_{\mathsf{LF}[i]} &<_{\text{lex}}\ & \mathcal{M}_{\mathsf{LF}[j]} \\ \Leftrightarrow\ & \mathsf{LF}[i] &<\ & \mathsf{LF}[j] \end{aligned}$$

Observation **??** allows to compute the LF-mapping without using the suffix array as the i-th occurrence of a character α in L is mapped to the i-th occurrence of α in F.

Example 10. T = `mississippi$`. The L-to-F mapping preserves the relative order of indices of matrix rows that end with the same character. The increasing sequence of *all* indices $3 < 4 < 9 < 10$ of rows that end with s is mapped to the increasing and *contiguous* sequence $9 < 10 < 11 < 12$.

F		L	i	LF(i)
`$`	`mississipp`	`i`	1	2
`i`	`$mississip`	`p`	2	7
`i`	`ppi$missis`	$\boxed{s}$	3	**9**
`i`	`ssippi$mis`	$\boxed{s}$	4	**10**
`i`	`ssissippi$`	`m`	5	6
`m`	`ississippi`	`$`	6	1
`p`	`i$mississi`	`p`	7	8
`p`	`pi$mississ`	`i`	8	3
$\boxed{s}$	`ippi$missi`	$\boxed{s}$	9	**11**
$\boxed{s}$	`issippi$mi`	$\boxed{s}$	10	**12**
$\boxed{s}$	`sippi$miss`	`i`	11	4
$\boxed{s}$	`sissippi$m`	`i`	12	5

Definition 11. Let T be a text of length n over an alphabet Σ and L the BWT of T.

- Let $C : \Sigma \rightarrow [0..n]$ be a function that maps a character $c \in \Sigma$ to the total number of occurrences in T of the characters which are alphabetically smaller than c.
- Let $\mathsf{Occ} : \Sigma \times [1..n] \rightarrow [1..n]$ be a function that maps $(c, k) \in \Sigma \times [1..n]$ to the number of occurrences of c in the prefix $L[1..k]$ of the transformed text L.

Theorem 12. *For the L-to-F mapping* LF *of a text T holds:*

$$\mathsf{LF}(i) = C(L[i]) + \mathsf{Occ}(L[i], i)$$

Proof: Let $\alpha = L[i]$. Of all occurrences of the character α in L, $\mathsf{Occ}(L[i], i)$ gives index of the occurrence at position i starting counting from 1. $C(L[i]) + j$ is the position of the j-th occurrence of α in F starting counting from 1. With $j = \mathsf{Occ}(L[i], i)$ the j-th occurrence of α in L is mapped to the j-th occurrence of α in F.

How can we back-transform L to T? With the L-to-F mapping we reconstruct T from right to left by cyclic shifting by one to the right beginning with $T^{(n)}$ and extracting the first characters. That can be done using the following properties:

- The last character of T is \$, whose only occurrence in F is $F[1]$, thus $\mathcal{M}_1 = T^{(n)}$ is T shifted by one to the right.
- $\mathcal{M}_{\mathsf{LF}(1)} = T^{(n)^{(n)}} = T^{(n-1)}$ is T shifted by 2 to the right. Therefore $F[\mathsf{LF}(1)]$ is the character before the last one in T.
- The character $T[n - i]$ is $F[\underbrace{\mathsf{LF}(\mathsf{LF}(\ldots \mathsf{LF}(1) \ldots))}_{i \text{ times } \mathsf{LF}}]$.

For the reverse transform we need F and the functions C and Occ. C and F can be obtained by bucket sorting L. LF only uses values $\mathsf{Occ}(L[i], i)$ which can be precomputed in an array of size n by a sequential scan over L. The pseudo-code for the reverse transform of $L = T^{\mathrm{bwt}}$ is given in algorithm **reverse_transform**. We replaced F by L using $F[\mathsf{LF}(i)] = L[i]$ and $F[1] = \$$.

Example 13. Reverse transform L = `ipssm$pissii` of length $n = 12$ over the alphabet $\Sigma = \{\$, \mathtt{i}, \mathtt{m}, \mathtt{p}, \mathtt{s}\}$. First, we count the number of occurrences n_α of every character $\alpha \in \Sigma$ in L and compute the partial sums $C(\alpha) = \sum_{\beta<\alpha} n_\beta$ of characters smaller than α to obtain C.

$\alpha \in \Sigma$	`$`	`i`	`m`	`p`	`s`
n_α	1	4	1	2	4
$C(\alpha)$	0	1	5	6	8

```
(1) // reverse_transform(L,Occ,C)
(2) $i = 1, j = n, c = \$$;
(3) while ($j > 0$) do
(4)         $T[j] = c$;
(5)         $c = L[i]$;
(6)         $i = C(c) + \mathrm{Occ}(c, i)$;
(7)         $j = j - 1$;
(8) od
(9) return $T$;
```

F is the concatenated sequence of runs of n_α many characters α in increasing order:

$$\begin{aligned} F &= \$^{n_\$} \cdot \mathtt{i}^{n_\mathtt{i}} \cdot \mathtt{m}^{n_\mathtt{m}} \cdot \mathtt{p}^{n_\mathtt{p}} \cdot \mathtt{s}^{n_\mathtt{s}} \\ &= \mathtt{\$iiiimppssss} \end{aligned}$$

For every i we precompute $\mathrm{Occ}(L[i], i)$ by sequentially scanning L and counting the number of occurrences of $L[i]$ in L up to position i. That can be done during the first run, where we determine the values n_α.

i	1	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{Occ}(L[i], i)$	1	1	1	2	1	1	2	2	3	4	3	4

Begin in row $i = 1$:
Extract the character $T[n] = F[1] = \$$
$T = \ldots\ldots\ldots\ldots\ldots\ldots\,\mathtt{\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[1]) + \mathrm{Occ}(L[1], 1) = 1 + 1 = 2$
Extract the character $T[n-1] = F[2] = \mathtt{i}$
$T = \ldots\ldots\ldots\ldots\ldots\,\mathtt{i\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[2]) + \mathrm{Occ}(L[2], 2) = 6 + 1 = 7$
Extract the character $T[n-2] = F[7] = \mathtt{p}$
$T = \ldots\ldots\ldots\ldots\ldots\,\mathtt{pi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[7]) + \mathrm{Occ}(L[7], 7) = 6 + 2 = 8$
Extract the character $T[n-3] = F[8] = \mathtt{p}$
$T = \ldots\ldots\ldots\ldots\,\mathtt{ppi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[8]) + \mathrm{Occ}(L[8], 8) = 1 + 2 = 3$
Extract the character $T[n-4] = F[3] = \mathtt{i}$
$T = \ldots\ldots\ldots\ldots\,\mathtt{ippi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[3]) + \mathrm{Occ}(L[3], 3) = 8 + 1 = 9$
Extract the character $T[n-5] = F[9] = \mathtt{s}$
$T = \ldots\ldots\ldots\,\mathtt{sippi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[9]) + \mathrm{Occ}(L[9], 9) = 8 + 3 = 11$
Extract the character $T[n-6] = F[11] = \mathtt{s}$
$T = \ldots\ldots\ldots\,\mathtt{ssippi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[11]) + \mathrm{Occ}(L[11], 11) = 1 + 3 = 4$
Extract the character $T[n-7] = F[4] = \mathtt{i}$
$T = \ldots\ldots\,\mathtt{issippi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[4]) + \mathrm{Occ}(L[4], 4) = 8 + 2 = 10$
Extract the character $T[n-8] = F[10] = \mathtt{s}$
$T = \ldots\,\mathtt{sissippi\$}$

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[10]) + \mathsf{Occ}(L[10], 10) = 8 + 4 = 12$
Extract the character $T[n-9] = F[12] =$ `s`
$T =$ `. . ssissippi$`

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[12]) + \mathsf{Occ}(L[12], 12) = 1 + 4 = 5$
Extract the character $T[n-10] = F[5] =$ `i`
$T =$ `. ississippi$`

Proceed with row $i = \mathbf{LF}(i)$:
$i = C(L[5]) + \mathsf{Occ}(L[5], 5) = 5 + 1 = 6$
Extract the character $T[1] = F[6] =$ `m`
$T =$ `mississippi$`

10.7 Backward search

This exposition has been developed by David Weese. It is based on the following sources, which are all recommended reading:

1. P. Ferragina, G. Manzini (2000) *Opportunistic data structures with applications*, Proceedings of the 41st IEEE Symposium on Foundations of Computer Science
2. P. Ferragina, G. Manzini (2001) *An experimental study of an opportunistic index*, Proceedings of the 12th ACM-SIAM Symposium on Discrete Algorithms, pp. 269-278

For a pattern $P = P[1..m]$ we want to count the number of occurrences in a text $T = T[1..m]$ given its Burrows-Wheeler transform $L = T^{\mathrm{bwt}}$. If we would have access to the conceptual matrix $\mathcal{M}$ we could conduct a binary search like in a suffix array. However, as we have direct access to only F and L we need a different approach.

Ferragina and Manzini proposed a backward search algorithm that searches the pattern from right to left by matching growing suffixes of P. It maintains an interval of matches and transforms the interval of matches of a suffix of P to an interval of the suffix which is one character longer. At the end, the length of the interval of the whole pattern P equals the number of occurrences of P.

Occurrences can be represented as intervals due to the following observation:

Observation 14. For every suffix of $P[j..m]$ of P the matrix rows $\mathcal{M}_i$ with prefix $P[j..m]$ form a contiguous block. Thus there are $a, b \in [1..m]$ such that $i \in [a..b] \Leftrightarrow \mathcal{M}_i[1..m-j+1] = P[j..m]$.

Proof: That is direct consequence of the lexicographical sorting of cyclic shifts in $\mathcal{M}$. Note that $[a..b] = \emptyset$ for $a > b$.

Consider $[a_j..b_j]$ to be the interval of matrix rows beginning with $P_j = P[j..m]$. In that interval we search cyclic shifts whose matching prefix is preceded by $c = P[j-1]$ in the cyclic text, i. e. matrix rows that end with c. If we shift these rows by 1 to the right they begin with $P[j-1..m]$ and determine the next interval $[a_{j-1}..b_{j-1}]$.

Matrix rows that end with c are occurrences of c in $L[a_j..b_j]$. The L-to-F mapping of the first and last occurrence yields a_{j-1} and b_{j-1} (rank preservation, lemma **??**).

The L-to-F mapping maps the i-th c in L to the i-th c in F. How to determine i for the first and last c in $L[a_j..b_j]$ without scanning ?

In L are $\mathsf{Occ}(c, a_j - 1)$ occurrences of c before the first occurrence in $L[a_j..b_j]$, hence:

$$i_f = \mathsf{Occ}(c, a_j - 1) + 1$$

i_l the number of the last occurrence of c in $L[a_j..b_j]$ equals the number of occurrences of c in $L[1..b_j]$:

$$i_l = \mathsf{Occ}(c, b_j)$$

Now, we can determine a_{j-1} and b_{j-1} by:

$$\begin{aligned} a_{j-1} &= C(c) + \mathsf{Occ}(c, a_j - 1) + 1 \\ b_{j-1} &= C(c) + \mathsf{Occ}(c, b_j) \end{aligned}$$

```
// count(P[1..m])
$i = m, a = 1, b = n$;
while (($a < b$) ∧ ($i \geq 1$)) do
        $c = P[i]$;
        $a = C(c) + \mathrm{Occ}(c, a-1) + 1$;
        $b = C(c) + \mathrm{Occ}(c, b)$;
        $i = i - 1$;
od
if ($b < a$) then return "not found";
            else return "found ($b - a + 1$) occurrences";
fi
```

Algorithm **count** computes the number of occurrences of $P[1..m]$ in $T[1..n]$:

Example 15. T = `mississippi$`. Search P = `ssi`.

Initialization: We begin with the empty suffix ϵ which is a prefix of every suffix, hence we initialize $a_{m+1} = 1$ and $b_{m+1} = n$.

$$\begin{array}{llll}
 & \mathrm{F} & & \mathrm{L} \\
a_4 \longrightarrow & \texttt{\$} & \texttt{mississipp} & \texttt{i} \\
 & \texttt{i} & \texttt{\$mississip} & \texttt{p} \\
 & \texttt{i} & \texttt{ppi\$missis} & \texttt{s} \\
 & \texttt{i} & \texttt{ssippi\$mis} & \texttt{s} \\
 & \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\
 & \texttt{m} & \texttt{ississippi} & \texttt{\$} \\
 & \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\
 & \texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\
 & \texttt{s} & \texttt{ippi\$missi} & \texttt{s} \\
 & \texttt{s} & \texttt{issippi\$mi} & \texttt{s} \\
 & \texttt{s} & \texttt{sippi\$miss} & \texttt{i} \\
b_4 \longrightarrow & \texttt{s} & \texttt{sissippi\$m} & \texttt{i}
\end{array}$$

$a_4 = 1$
$b_4 = 12$

Searching P_3: From all matrix rows we search those beginning with the last pattern character $P[3]$ = `i`. From $\mathrm{Occ}(x, 0) = 0$ and $\mathrm{Occ}(x, n) = n_x$ follows $a_m = C(x) + 1$ and $b_m = C(x + 1)$.

$$\begin{array}{llll}
 & \mathrm{F} & & \mathrm{L} \\
a_4 \longrightarrow & \texttt{\$} & \texttt{mississipp} & \boxed{\texttt{i}} \\
 & \texttt{i} & \texttt{\$mississip} & \texttt{p} \\
 & \texttt{i} & \texttt{ppi\$missis} & \texttt{s} \\
 & \texttt{i} & \texttt{ssippi\$mis} & \texttt{s} \\
 & \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\
 & \texttt{m} & \texttt{ississippi} & \texttt{\$} \\
 & \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\
 & \texttt{p} & \texttt{pi\$mississ} & \boxed{\texttt{i}} \\
 & \texttt{s} & \texttt{ippi\$missi} & \texttt{s} \\
 & \texttt{s} & \texttt{issippi\$mi} & \texttt{s} \\
 & \texttt{s} & \texttt{sippi\$miss} & \boxed{\texttt{i}} \\
b_4 \longrightarrow & \texttt{s} & \texttt{sissippi\$m} & \boxed{\texttt{i}}
\end{array}
\quad \Rightarrow \quad
\begin{array}{lll}
\mathrm{F} & & \mathrm{L} \\
\texttt{\$} & \texttt{mississipp} & \texttt{i} \\
\boxed{\texttt{i}} & \texttt{\$mississip} & \texttt{p} \\
\boxed{\texttt{i}} & \texttt{ppi\$missis} & \texttt{s} \\
\boxed{\texttt{i}} & \texttt{ssippi\$mis} & \texttt{s} \\
\boxed{\texttt{i}} & \texttt{ssissippi\$} & \texttt{m} \\
\texttt{m} & \texttt{ississippi} & \texttt{\$} \\
\texttt{p} & \texttt{i\$mississi} & \texttt{p} \\
\texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\
\texttt{s} & \texttt{ippi\$missi} & \texttt{s} \\
\texttt{s} & \texttt{issippi\$mi} & \texttt{s} \\
\texttt{s} & \texttt{sippi\$miss} & \texttt{i} \\
\texttt{s} & \texttt{sissippi\$m} & \texttt{i}
\end{array}$$

$a_4 = 1$
$b_4 = 12$

$a_3 = C(\texttt{i}) + 1 = 1 + 1$
$b_3 = C(\texttt{i}) + n_{\texttt{i}} = 1 + 4$

Searching P_2: From all rows beginning with P_3 we search those beginning with $P[2]$ = `s`. In L we count the `s`'s in the part before the interval (=0) and including the interval (=2) to L-to-F map the first and last `s` in the interval.

$$\begin{array}{rccc} & \text{F} & & \text{L} \\ & \texttt{\$} & \texttt{mississipp} & \texttt{i} \\ a_3 \rightarrow & \texttt{i} & \texttt{\$mississip} & \texttt{p} \\ & \texttt{i} & \texttt{ppi\$missis} & \boxed{\texttt{s}} \\ & \texttt{i} & \texttt{ssippi\$mis} & \boxed{\texttt{s}} \\ b_3 \rightarrow & \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\ & \texttt{m} & \texttt{ississippi} & \texttt{\$} \\ & \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\ & \texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\ & \texttt{s} & \texttt{ippi\$missi} & \texttt{s} \\ & \texttt{s} & \texttt{issippi\$mi} & \texttt{s} \\ & \texttt{s} & \texttt{sippi\$miss} & \texttt{i} \\ & \texttt{s} & \texttt{sissippi\$m} & \texttt{i} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \text{F} & & \text{L} \\ \texttt{\$} & \texttt{mississipp} & \texttt{i} \\ \texttt{i} & \texttt{\$mississip} & \texttt{p} \\ \texttt{i} & \texttt{ppi\$missis} & \texttt{s} \\ \texttt{i} & \texttt{ssippi\$mis} & \texttt{s} \\ \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\ \texttt{m} & \texttt{ississippi} & \texttt{\$} \\ \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\ \texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\ \boxed{\texttt{s}} & \texttt{ippi\$missi} & \texttt{s} \\ \boxed{\texttt{s}} & \texttt{issippi\$mi} & \texttt{s} \\ \texttt{s} & \texttt{sippi\$miss} & \texttt{i} \\ \texttt{s} & \texttt{sissippi\$m} & \texttt{i} \end{array}$$

$a_3 = 2$
$b_3 = 5$

$a_2 = C(\mathsf{s}) + \mathsf{Occ}(\mathsf{s}, 1) + 1 = 8 + 0 + 1$
$b_2 = C(\mathsf{s}) + \mathsf{Occ}(\mathsf{s}, 5) = 8 + 2$

Searching P_1: From all matrix rows beginning with P_2 we search those beginning with $P[1] = \mathsf{s}$.

$$\begin{array}{rccc} & \text{F} & & \text{L} \\ & \texttt{\$} & \texttt{mississipp} & \texttt{i} \\ & \texttt{i} & \texttt{\$mississip} & \texttt{p} \\ & \texttt{i} & \texttt{ppi\$missis} & \texttt{s} \\ & \texttt{i} & \texttt{ssippi\$mis} & \texttt{s} \\ & \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\ & \texttt{m} & \texttt{ississippi} & \texttt{\$} \\ & \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\ & \texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\ a_2 \rightarrow & \texttt{s} & \texttt{ippi\$missi} & \boxed{\texttt{s}} \\ b_2 \rightarrow & \texttt{s} & \texttt{issippi\$mi} & \boxed{\texttt{s}} \\ & \texttt{s} & \texttt{sippi\$miss} & \texttt{i} \\ & \texttt{s} & \texttt{sissippi\$m} & \texttt{i} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \text{F} & & \text{L} \\ \texttt{\$} & \texttt{mississipp} & \texttt{i} \\ \texttt{i} & \texttt{\$mississip} & \texttt{p} \\ \texttt{i} & \texttt{ppi\$missis} & \texttt{s} \\ \texttt{i} & \texttt{ssippi\$mis} & \texttt{s} \\ \texttt{i} & \texttt{ssissippi\$} & \texttt{m} \\ \texttt{m} & \texttt{ississippi} & \texttt{\$} \\ \texttt{p} & \texttt{i\$mississi} & \texttt{p} \\ \texttt{p} & \texttt{pi\$mississ} & \texttt{i} \\ \texttt{s} & \texttt{ippi\$missi} & \texttt{s} \\ \texttt{s} & \texttt{issippi\$mi} & \texttt{s} \\ \boxed{\texttt{s}} & \texttt{sippi\$miss} & \texttt{i} \\ \boxed{\texttt{s}} & \texttt{sissippi\$m} & \texttt{i} \end{array}$$

$a_2 = 9$
$b_2 = 10$

$a_1 = C(\mathsf{s}) + \mathsf{Occ}(\mathsf{s}, 8) + 1 = 8 + 2 + 1$
$b_1 = C(\mathsf{s}) + \mathsf{Occ}(\mathsf{s}, 10) = 8 + 4$

Found the interval for P: $[a_1..b_1]$ is the interval of matrix rows with prefix P, thus P has $b_1 - a_1 + 1 = 2$ occurrences in the text T.

$$\begin{array}{rl} & \texttt{\$mississippi} \\ & \texttt{i\$mississipp} \\ & \texttt{ippi\$mississ} \\ & \texttt{issippi\$miss} \\ & \texttt{ississippi\$m} \\ & \texttt{mississippi\$} \\ & \texttt{pi\$mississip} \\ & \texttt{ppi\$mississi} \\ & \texttt{sippi\$missis} \\ & \texttt{sissippi\$mis} \\ a_1 \rightarrow & \mathbf{\texttt{ssi}}\texttt{ppi\$missi} \\ b_1 \rightarrow & \mathbf{\texttt{ssi}}\texttt{ssippi\$mi} \end{array}$$

$a_1 = 11$
$b_1 = 12$

10.8 Locate matches

We have seen how to count occurrences of a pattern P in the text T, but how to obtain their location in the text? Algorithm **count** determines the indexes $a, a+1, \dots, b$ of matrix rows with prefix P. As cyclic shifts correspond to the suffixes of T, with a suffix array A of T we would be able to get the corresponding text position $pos(i)$ of the suffix in row i. It holds $pos(i) = A[i]$.

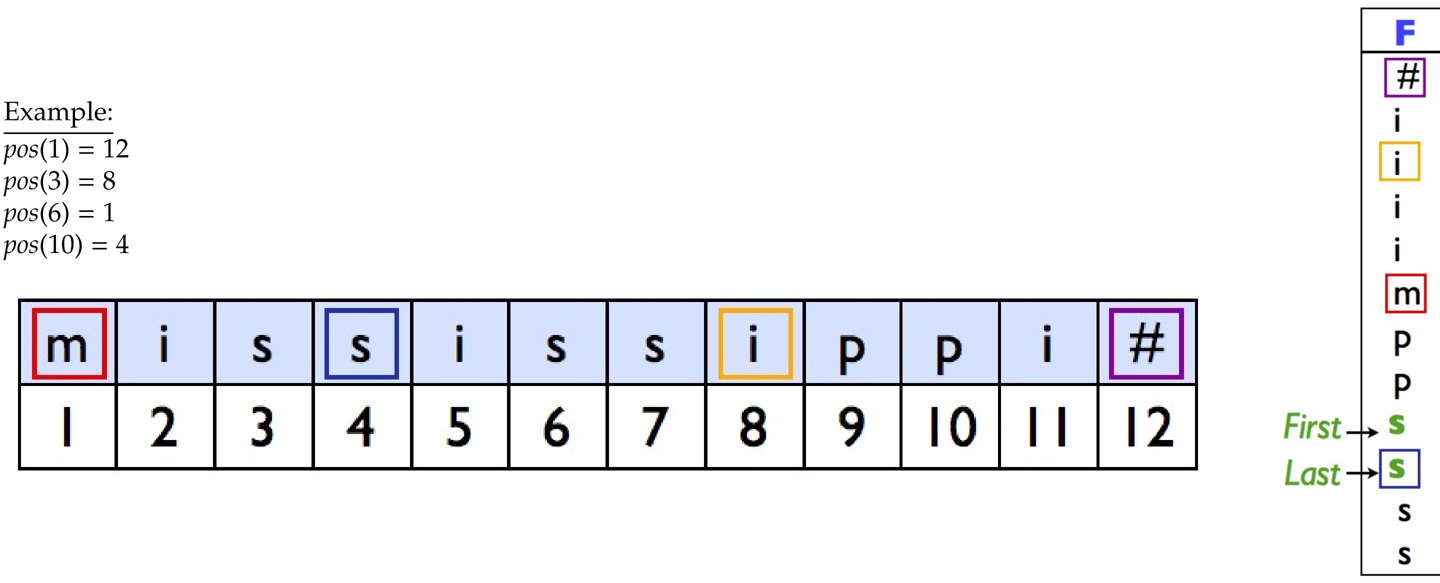
We will now see that it is not necessary to have the whole suffix array with $4n$ bytes of memory. It suffices to have a fraction of the suffix array available to compute $pos(i)$ for every $i \in [1..n]$.

The idea is as following. We logically mark a suitable subset of rows in the matrix. For the marked rows we explicitly store the start positions of the suffixes in the text. If i is marked, row $pos(i)$ is directly available. If i is not marked, the algorithm **locate** uses the L-to-F-mapping to find the row $i_1 = \mathsf{LF}(i)$ corresponding to the suffix $T[pos(i) - 1..n]$. This procedure is iterated v times until we reach a marked row i_v for which $pos(i_v)$ is available; then we set $pos(i) = pos(i_v) + v$.

This is a direct space-time trade-off.

Example 16.

Example:
$pos(1) = 12$
$pos(3) = 8$
$pos(6) = 1$
$pos(10) = 4$



m	i	s	s	i	s	s	i	p	p	i	#
1	2	3	4	5	6	7	8	9	10	11	12

Preprocessing. The position of every x-th letter in the text is marked and stored for the corresponding row in S.

For a row i, algorithm **locate** determines the location of the corresponding occurrence in $T[1..n]$.

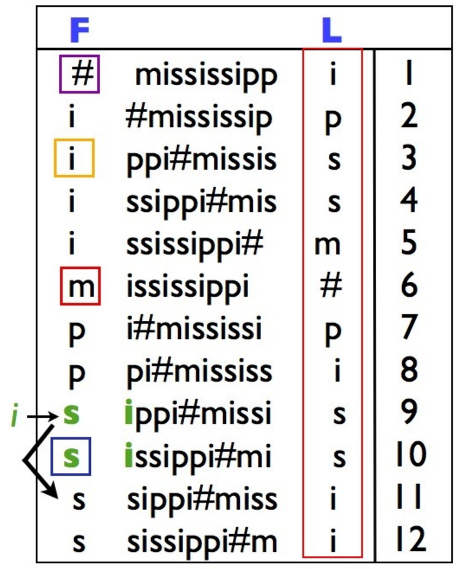
```
(1) // locate(i)
(2) $i' = i$
(3) $v = 0$;
(4) while (row $i'$ is not marked) do
(5)     $c = L[i']$;
(6)     $i' = C(c) + \mathsf{Occ}(c, i')$;
(7)     $v = v + 1$;
(8) od
(9) return $pos(i') + v$;
```

locate(i) is called for every $i = [a..b]$, where $[a..b]$ is interval of occurrences computed by **count**(P). We call the conjunction of both algorithms and their required data structures the *FM Index*.

In the following we give an example using the BWT of the text `mississippi#`, the thinned out suffix array and the interval $[a..b]$ resulting from the search of the pattern `si`.

For each $i = [9, 10]$ we have to its position in the text. **i = 9**

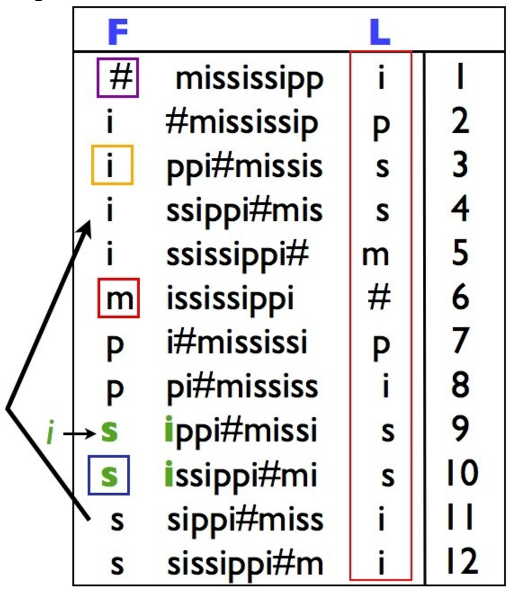
Step 1



F		L	
#	mississipp	i	1
i	#mississip	p	2
i	ppi#missis	s	3
i	ssippi#mis	s	4
i	ssissippi#	m	5
m	ississippi	#	6
p	i#mississi	p	7
p	pi#mississ	i	8
s	ippi#missi	s	9
s	issippi#mi	s	10
s	sippi#miss	i	11
s	sissippi#m	i	12

row 9 is not marked
→ L-to-F(9)=11
→ Look at row 11
v=1

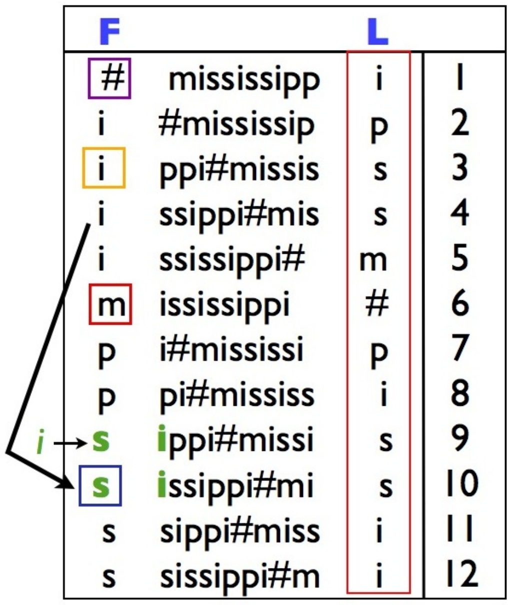
Step 2



F		L	
#	mississipp	i	1
i	#mississip	p	2
i	ppi#missis	s	3
i	ssippi#mis	s	4
i	ssissippi#	m	5
m	ississippi	#	6
p	i#mississi	p	7
p	pi#mississ	i	8
s	ippi#missi	s	9
s	issippi#mi	s	10
s	sippi#miss	i	11
s	sissippi#m	i	12

row 11 is not marked
→ L-to-F(11)=4
→ Look at row 4
v=2

Step 3



F		L	
#	mississipp	i	1
i	#mississip	p	2
i	ppi#missis	s	3
i	ssippi#mis	s	4
i	ssissippi#	m	5
m	ississippi	#	6
p	i#mississi	p	7
p	pi#mississ	i	8
s	ippi#missi	s	9
s	issippi#mi	s	10
s	sippi#miss	i	11
s	sissippi#m	i	12

row 4 is not marked
→ L-to-F(4)=10
→ Look at row 10
v=3

row 10 is marked
Calculation of the $pos(9)$:
$pos(9) = pos(10) + 3 = 4 + 3 = 7$

We saw how to avoid storing the complete suffix array when locating the text.

However, the table Occ is still quite big. It contains the number of occurrences for each character and each possible prefix of L needing $|\Sigma| \times n$ entries storing the number of occurrences.

One way to reduce the size of Occ is to store only every x-th index. The entries that are omitted can be reconstructed from stored entries at the cost of an increased run time, by simply counting in the BWT from the last stored position on.

Taken together, we can conduct an exact search in a text in time *linear* to the query size.

For example (for DNA) using the the text T (n bytes), the BWT (n bytes resp. $n/4$ bytes), the *Occ* table (e.g. $4 \cdot 4 \cdot n/32$ bytes when storing only every 32th entry) and a sampled suffix array (e.g. $4 \cdot n/8$ bytes when marking every 8th entry). In our example calculation we would need about $2.25n - 3n$ bytes.