Turing machines codes

- May assume

$$M = (Q, \{0,1\}, \{0,1,\#\}, \delta, q_1, \#, \{q_2\})$$

- Unary encoding

$$0 \mapsto 0, 1 \mapsto 00, \# \mapsto 000, L \mapsto 0, R \mapsto 00$$

- $\delta(q_i, X) = (q_j, Y, R)$ encoded by

$$0^i 1 \underbrace{0 \dots 0}_{X} 1 0^j 1 \underbrace{0 \dots 0}_{Y} 1 \underbrace{0 \dots 0}_{R}$$

- δ encoded by

$$111 \, \mathtt{code}_1 \, 11 \, \mathtt{code}_2 \, 11 \dots 11 \, \mathtt{code}_r \, 111$$

- Encoding of Turing machine M denoted by $\langle M \rangle$.

Numbering of Turing machines

- **Lemma.** There exists a Turing machine that generates the natural numbers in binary encoding.
- **Lemma.** There exists a Turing machine *Gen* that generates the binary encodings of all Turing machines.
- **Proposition.** The language of Turing machine codes is recursive.
- **Corollary.** There exist a bijection between the set of natural numbers, Turing machine codes and Turing machines.

Gen $\longrightarrow$ [Equality test + counter] $\longrightarrow$ number n

$M \longrightarrow \langle M \rangle \longrightarrow$ [Equality test + counter]

Diagonalization

- Let w_i be the i-th word in $\{0,1\}^*$ and M_j the j-th Turing machine.
- Table T with $t_{ij} = \begin{cases} 1, & \text{if } w_i \in L(M_j) \\ 0, & \text{if } w_i \notin L(M_j) \end{cases}$

$i \downarrow$ \ $j \longrightarrow$	1	2	3	4	...
1	0	1	1	0	...
2	1	1	0	1	...
3	0	0	1	0	...
⋮	⋮	⋮	⋮	⋮	

- *Diagonal language* $L_d = \{w_i \in \{0,1\}^* \mid w_i \notin L(M_i)\}$.
- **Theorem.** L_d is not recursively enumerable.
- *Proof:* Suppose $L_d = L(M_k)$, for some $k \in \mathbb{N}$. Then

$$w_k \in L_d \Leftrightarrow w_k \notin L(M_k),$$

contradicting $L_d = L(M_k)$.

Universal language

- $\langle M, w \rangle$: encoding $\langle M \rangle$ of M concatenated with $w \in \{0,1\}^*$.
- *Universal language*

$$L_u = \{\langle M, w \rangle \mid M \text{ accepts } w\}$$

- **Theorem.** L_u is recursively enumerable.
- A Turing machine U accepting L_u is called *universal Turing machine.*
- **Theorem** (Turing 1936). L_u is not recursive.

Decision problems

- Decision problems are problems with answer either yes or no.
- Associate with a language $L \subseteq \Sigma^*$ the decision problem D_L

  Input: $w \in \Sigma^*$

  Output: $\begin{cases} \text{yes}, & \text{if } w \in L \\ \text{no}, & \text{if } w \notin L \end{cases}$

  and vice versa.
- D_L is *decidable* (resp. *semi-decidable*) if L is recursive (resp. recursively enumerable).
- D_L is *undecidable* if L is not recursive.

Reductions

- A *many-one reduction* of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a computable function $f : \Sigma_1^* \to \Sigma_2^*$ with $w \in L_1 \Leftrightarrow f(w) \in L_2$.
- **Proposition.** If L_1 is many-one reducible to L_2, then
  1. L_1 is decidable if L_2 is decidable.
  2. L_2 is undecidable if L_1 is undecidable.

Post's correspondence problem

- Given pairs of words

$$(v_1, w_1), (v_2, w_2), \dots, (v_k, w_k)$$

  over an alphabet Σ, does there exist a sequence of integers $i_1, \dots, i_m, m \geq 1$, such that

$$v_{i_1}, \dots, v_{i_m} = w_{i_1}, \dots, w_{i_m}.$$

- *Example*

i	v_i	w_i
1	1	111
2	10111	10
3	10	0

$\Rightarrow v_2 v_1 v_1 v_3 = w_2 w_1 w_1 w_3 = 101111110$

- **Theorem** (Post 1946). Post's correspondence problem is undecidable.