Non-deterministic Turing machines

- *Next move relation:*
$$\delta \subseteq (Q \times \Gamma) \times (Q \times \Gamma \times \{L, R\})$$
- $L(M)$ = set of words $w \in \Sigma^*$ for which *there exists* a sequence of moves accepting w.
- **Proposition.** If L is accepted by a non-deterministic Turing machine M_1, then L is accepted by some deterministic machine M_2.

Time complexity

- M a (deterministic) Turing machine that halts on all inputs.
- Time complexity function $T_M : \mathbb{N} \to \mathbb{N}$
$$\begin{aligned} T_M(n) \quad = \quad & \max\{m \mid \exists w \in \Sigma^*, |w| = n \text{ such that the computation} \\ & \text{of } M \text{ on } w \text{ takes } m \text{ moves}\} \end{aligned}$$

  (assume numbers are coded in binary format)
- A Turing machine is *polynomial* if there exists a polynomial $p(n)$ with $T_M(n) \leq p(n)$, for all $n \in \mathbb{N}$.
- The *complexity class P* is the class of languages decided by a polynomial Turing machine.

Time complexity of non-deterministic Turing machines

- M non-deterministic Turing machine
- The running time of M on $w \in \Sigma^*$ is
  - the length of a shortest sequence of moves accepting w if $w \in L(M)$
  - 1, if $w \notin L(M)$
- $T_M(n) = \max\{m \mid \exists w \in \Sigma^*, |w| = n$ such that the running time of M on w is $m\}$
- The *complexity class NP* is the class of languages accepted by a polynomial non-deterministic Turing machine.

Deciding languages in NP

Theorem. If $L \in NP$, then there exists a deterministic Turing machine M and a polynomial $p(n)$ such that

- M decides L and
- $T_M(n) \leq 2^{p(n)}$, for all $n \in \mathbb{N}$.

Proof: Suppose L is accepted by a non-deterministic machine M_{nd} whose running time is bounded by the polynomial $q(n)$.

To decide whether $w \in L$, the machine M will

1. determine the length n of w and compute $q(n)$.
2. simulate all executions of M_{nd} of length at most $q(n)$. If the maximum number of choices of M_{nd} in one step is r, there are at most $r^{q(n)}$ such executions.

3. if one of the simulated executions accepts w, then M accepts w, otherwise M rejects w.

The overall complexity is bounded by $r^{q(n)} \cdot q'(n) = O(2^{p(n)})$, for some polynomial $p(n)$.

An alternative characterization of NP

- **Proposition.** $L \in NP$ if there exists $L' \in P$ and a polynomial $p(n)$ such that for all $w \in \Sigma^*$:

$$w \in L \;\Leftrightarrow\; \exists v \in (\Sigma')^* : |v| \leq p(|w|) \text{ and } (w, v) \in L'$$

- Informally, a problem is in NP if it can be solved non-deterministically in the following way:
  1. guess a solution/certificate v of polynomial length,
  2. check in polynomial time whether v has the desired property.

Propositional satisfiability

- *Satisfiability problem SAT*

  Instance: A formula F in propositional logic with variables $x_1, \ldots, x_n$.

  Question: Is F satisfiable, i.e., does there exist an assignment $I : \{x_1, \ldots, x_n\} \to \{0, 1\}$ making the formula true ?

- Trying all possible assignments would require exponential time.
- Guessing an assignment I and checking whether it satisfies F can be done in (non-deterministic) polynomial time. Thus:
- **Proposition.** SAT is in NP.

Polynomial reductions

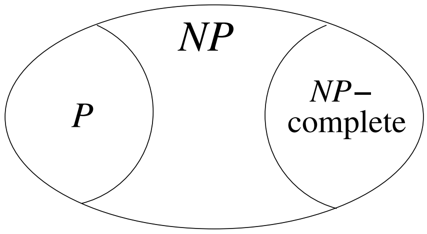
- A *polynomial reduction* of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a polynomially computable function $f : \Sigma_1^* \to \Sigma_2^*$ with $w \in L_1 \Leftrightarrow f(w) \in L_2$.
- **Proposition.** If L_1 is polynomially reducible to L_2, then
  1. $L_1 \in P$ if $L_2 \in P$ and $L_1 \in NP$ if $L_2 \in NP$
  2. $L_2 \notin P$ if $L_1 \notin P$ and $L_2 \notin NP$ if $L_1 \notin NP$.
- L_1 and L_2 are *polynomially equivalent* if they are polynomially reducible to each other.

NP-complete problems

- A language $L \subseteq \Sigma^*$ is *NP-complete* if
  1. $L \in NP$
  2. Any $L' \in NP$ is polynomially reducible to L.
- **Proposition.** If L is NP-complete and $L \in P$, then $P = NP$.

- **Corollary.** If L is NP-complete and $P \neq NP$, then there exists no polynomial algorithm for L.

Structure of the class NP



Fundamental open problem: $P \neq NP$ **?**

Proving NP-completeness

- **Theorem** (Cook 1971). SAT is *NP*-complete.
- **Proposition.** L is *NP*-complete if
  1. $L \in NP$
  2. there exists an *NP*-complete problem L' that is polynomially reducible to L.
- INDEPENDENT SET

  Instance: Graph $G = (V, E)$ and $k \in \mathbb{N}, k \leq |V|$.
  Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and no two vertices in V are joined by an edge in E ?

Reducing 3SAT to INDEPENDENT SET

- Let F be a conjunction of n *clauses* of length 3, i.e., a disjunction of 3 propositional variables or their negation.
- Construct a graph G with $3n$ vertices that correspond to the variables in F.
- For any clause in F, connect by three edges the corresponding vertices in G.
- Connect all pairs of vertices corresponding to a variable x and its negation $\neg x$.
- F is satisfiable if and only if G contains an independent set of size n.

Solving numerical constraints

Satisfiability	over $\mathbb{Q}$	over $\mathbb{Z}$	over $\mathbb{N}$
Linear equations	polynomial	polynomial	*NP*-complete
Linear inequalities	polynomial	*NP*-complete	*NP*-complete

Satisfiability	over $\mathbb{R}$	over $\mathbb{Z}$
Linear constraints	polynomial	*NP*-complete
Nonlinear constraints	decidable	undecidable

NP-hard problems

- *Decision problem:* solution is either yes or no
- Example: Traveling salesman decision problem:
  Given a network of cities, distances, and a number B, does there exist a tour with length $\leq$ B?
- *Search problem:* find an object with required properties
- Example: Traveling salesman optimization problem:
  Given a network of cities and distances, find a shortest tour.
- Decision problem *NP*-complete $\Rightarrow$ search problem *NP*-hard
- *NP-hard problems:* at least as hard as *NP*-complete problems

Graph theoretical problems

- Shortest path *polynomial*
- Traveling salesman *NP-hard*
- Minimum spanning tree *polynomial*
- Steiner tree *NP-hard*