Graph Algorithms

I. Shortest paths

- $D = (V, A)$ directed graph, $s, t \in V$.
- A *walk* is a sequence $P = (v_0, a_1, v_1, \dots, a_k, v_k), k \geq 0$, where a_i is an arc from v_{i-1} to v_i, for $i = 1, \dots, k$.
- P is a *path*, if $v_0, \dots, v_k$ are all different.
- If $s = v_0$ and $t = v_k$, P is a *s-t walk* resp. *s-t path* of *length k* (i.e., each arc has length 1).
- The *distance* from s to t is the minimum length of any s-t path (and $+\infty$ if no s-t path exists).

Shortest paths with unit lengths

Algorithm (Breadth-first search)

Initialization: $V_0 = \{s\}$
Iteration: $V_{i+1} = \{v \in V \setminus (V_0 \cup V_1 \cup \dots \cup V_i) \mid (u, v) \in A,$ for some $u \in V_i\}$,
until $V_{i+1} = \emptyset$.

Running time: $O(|A|)$

- V_i is the set of nodes with distance i from s.
- The algorithm computes shortest paths from s to all reachable nodes.
- Can be described by a directed tree $T = (V', A')$ with root s such that each u-v path in T is a shortest s-t path in D.

Shortest paths with non-negative lengths

- Length function $l : A \rightarrow \mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\}$
- For a walk $P = (v_0, a_1, v_1, \dots, a_k, v_k)$ define $l(P) = \sum_{i=1}^{k} l(a_i)$.

Algorithm (Dijkstra 1959)

Initialization: $U = V, f(s) = 0, f(v) = \infty$, for $v \in V \setminus \{s\}$
Iteration: Find $u \in U$ with $f(u) = \min\{f(v) \mid v \in U\}$.
For all $a = (u, v) \in A$ with $f(v) > f(u) + l(a)$ let $f(v) = f(u) + l(a)$.
Let $U \leftarrow U \setminus \{u\}$, until $U = \emptyset$.

Upon termination, $f(v)$ gives the length of a shortest path from s to v.

Running time: $O(|V|^2)$ (can be improved to $O(|A| + |V| \log |V|)$.)

Application: Longest common subsequence

- Sequences $a = a_1, \dots, a_m$ and $b = b_1, \dots, b_n$
- Find the longest common subsequence of a and b (obtained by removing symbols in a or b).

Modeling as a shortest path problem

- Grid graph with nodes $(i, j), 0 \le i \le m, 0 \le j \le n$.
- Horizontal and vertical arcs of length 1.
- Diagonal arcs $((i-1, j-1), (i, j))$ of length 0, if $a_i = b_j$.

The diagonal arcs on a shortest path from $(0, 0)$ to (m, n) define a longest common subsequence.

Circuits of negative length

- Consider arbitrary length functions $l : A \to \mathbb{Q}$.
- A *directed circuit* is a walk $P = (v_0, a_1, v_1, \dots, a_k, v_k)$ with $k \ge 1$ and $v_0 = v_k$ such that $v_1, \dots, v_k$ and $a_1, \dots, a_k$ are all different.
- If $D = (V, A)$ contains a directed circuit of negative length, there exist s-t walks of arbitrary small negative length.

Proposition
Let $D = (V, A)$ be a directed graph without circuits of negative length.
For any $s, t \in V$ for which there exists at least one s-t walk, there exists a shortest s-t walk, which is a path.

Shortest paths with arbitrary lengths

$D = (V, A), n = |V|, l : A \to \mathbb{Q}$.

Algorithm (Bellman-Ford 1956/58)

Compute $f_0, \dots, f_n : V \to \mathbb{R} \cup \{\infty\}$ in the following way:

Initialization: $f_0(s) = 0$, $f_0(v) = \infty$, for $v \in V \setminus \{s\}$
Iteration: For $k = 1, \dots, n$ and all $v \in V$:
$$f_k(v) = \min\{f_{k-1}(v), \min_{(u,v)\in A}(f_{k-1}(u) + l(u, v))\}$$

Running time: $O(|V||A|)$

Properties

- For each $k = 0, \dots, n$ and each $v \in V$:
$$f_k(v) = \min\{l(P) \mid P \text{ is an } s\text{-}v \text{ walk traversing at most } k \text{ arcs}\}$$
(by induction)
- If D contains no circuits of negative length, $f_{n-1}(v)$ is the length of a shortest path from s to v.

Finding an explicit shortest path

- When computing $f_0, \dots, f_n$ determine a predecessor function $p : V \to V$ by setting $p(v) = u$ whenever $f_{k+1}(v) = f_k(u) + l(u, v)$.
- At termination, $v, p(v), p(p(v)), \dots, s$ gives the reverse of a shortest s-v path.

Theorem
Given $D = (V, A), s, t \in V$ and $l : A \to \mathbb{Q}$ such that D contains no circuit of negative length, a shortest s-t path can be found in time $O(|V||A|)$.

Remark
D contains a circuit of negative length reachable from s if and only if $f_n(v) \neq f_{n-1}(v)$, for some $v \in V$.

NP-completeness

For directed graphs containing circuits of negative length, the problem becomes NP-complete:

Theorem
The decision problem

> *Input:* Directed graph $D = (V, A)$, $s, t \in V$, $l : A \to \mathbb{Z}$, $L \in \mathbb{Z}$
> *Question:* Does there exist an s-t path P with $l(P) \leq L$?

is NP-complete.

Corollary
The shortest path problem with arbitrary lengths is NP-complete.
The longest path problem with non-negative lengths is NP-complete.

Application: Knapsack problem

- Knapsack, volume 8, 5 articles

Article i	Volume a_i	Value c_i
1	5	4
2	3	7
3	2	3
4	2	5
5	1	4

- Objective: Select articles fitting into the knapsack and maximizing the total value.

Possible models

- *Linear 0-1 model*

$$\max\{4x_1 + 7x_2 + 3x_3 + 5x_4 + 4x_5 \mid 5x_1 + 3x_2 + 2x_3 + 2x_4 + x_5 \leq 8, x_1, \ldots, x_5 \in \{0, 1\}\}$$

- *Shortest path model*
  - Directed graph with nodes $(i, x), 0 \leq i \leq 6,\ 0 \leq x \leq 8$.
  - Arcs from $(i-1, x)$ to (i, x) resp. $(i, x + a_i)$ of length 0 resp. $-c_i$, for $0 \leq i \leq 5$.
  - Arcs from $(5, x)$ to $(6, 8)$ of length 0, for $0 \leq x \leq 6$.
  - A shortest path from $(0, 0)$ to $(6, 8)$ gives an optimal solution.

$\rightsquigarrow$ *pseudo-polynomial algorithm*

II. Network flows

- *Network*
  - Directed graph $G = (V, E)$
  - *Source* $s \in V$, *sink* $t \in V$
  - *Edge capacities* $\mathrm{cap} : E \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$
- *Flow:* $f : E \to \mathbb{R}_+$ satisfying
  1. Flow conservation constraints
  $$\sum_{e:\mathrm{target}(e)=v} f(e) = \sum_{e:\mathrm{source}(e)=v} f(e),\ \text{for all } v \in V \setminus \{s, t\}$$
  2. Capacity constraints
  $$0 \leq f(e) \leq \mathrm{cap}(e),\ \text{for all } e \in E$$

Maximum flow problem

- *Excess* at node v: $\mathrm{excess}(v) = \sum_{e:\mathrm{target}(e)=v} f(e) - \sum_{e:\mathrm{source}(e)=v} f(e)$
- If f is a flow, then $\mathrm{excess}(v) = 0$, for all $v \in V \setminus \{s, t\}$.
- *Value* of a flow: $\mathrm{val}(f) \stackrel{\mathrm{def}}{=} \mathrm{excess}(t)$
- **Maximum flow problem:**
$$\max\{\mathrm{val}(f) \mid f \text{ is a flow in } G\}$$
- Can be seen as a linear programming problem.

Maximum flow problem (2)

Lemma
If f is a flow, then $\mathrm{excess}(t) = -\mathrm{excess}(s)$.

Proof: We have
$$\mathrm{excess}(s) + \mathrm{excess}(t) = \sum_{v \in V} \mathrm{excess}(v) = 0.$$

- First "=": $\mathrm{excess}(v) = 0$, for $v \in V \setminus \{s, t\}$
- Second "=": For any edge $e = (v, w)$, the flow through e appears twice in the sum, positively in $\mathrm{excess}(w)$ and negatively in $\mathrm{excess}(v)$.

Cuts

- A *cut* is a partition (S, T) of V, i.e., $T = V \setminus S$.
- (S, T) is an (s, t)-*cut* if $s \in S$ and $t \in T$.
- *Capacity* of the cut (S, T)
$$\mathrm{cap}(S, T) = \sum_{E \cap (S \times T)} \mathrm{cap}(e)$$
- A cut is *saturated by f* if $f(e) = \mathrm{cap}(e)$, for all $e \in E \cap (S \times T)$, and $f(e) = 0$, for all $e \in E \cap (T \times S)$.

Cuts (2)

Lemma
If f is a flow and (S, T) an (s, t)-cut, then

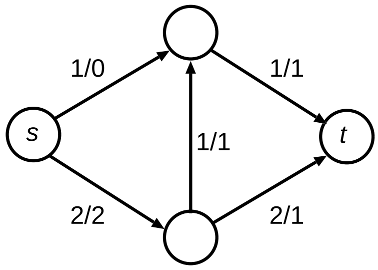
$$\operatorname{val}(f) = \sum_{e \in E \cap (S \times T)} f(e) - \sum_{e \in E \cap (T \times S)} f(e) \leq \operatorname{cap}(S, T).$$

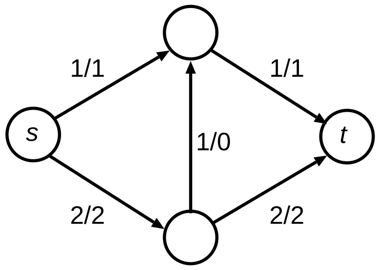
If S is saturated by f, then $\operatorname{val}(f) = \operatorname{cap}(S, T)$.

Proof: We have

$$\begin{aligned} \operatorname{val}(f) &= -\operatorname{excess}(s) = -\sum_{u \in S} \operatorname{excess}(u) = \sum_{e \in E \cap (S \times T)} f(e) - \sum_{e \in E \cap (T \times S)} f(e) \\ &\leq \sum_{e \in E \cap (S \times T)} \operatorname{cap}(e) = \operatorname{cap}(S) \end{aligned}$$

For a saturated cut, the inequality is an equality.





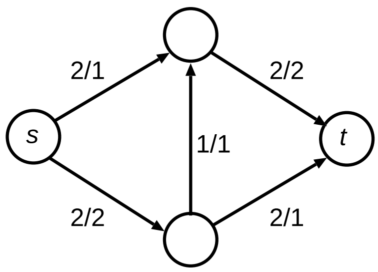
Remarks

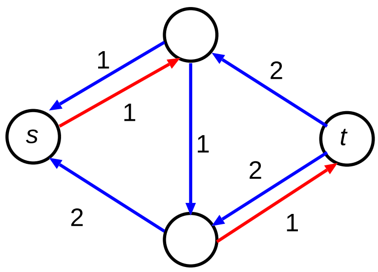
- A saturated cut proves the optimality of a flow.
- To show: for every maximal flow there is a saturated cut proving its optimality.

Residual network

The *residual network* G_f for a flow f in $G = (V, E)$ indicates the capacity unused by f. It is defined as follows:

- G_f has the same node set as G.
- For every edge $e = (v, w)$ in G, there are up to two edges e' and e'' in G_f:
  1. if $f(e) < \operatorname{cap}(e)$, there is an edge $e' = (v, w)$ in G_f with *residual capacity* $r(e') = \operatorname{cap}(e) - f(e)$.
  2. if $f(e) > 0$, there is an edge $e'' = (w, v)$ in G_f with residual capacity $r(e'') = f(e)$.





Theorem
Let f be an (s, t)-flow, let G_f be the residual network w.r.t. f, and let S be the set of all nodes reachable from s in G_f.

1. If $t \in S$, then f is not maximum.
2. If $t \notin S$, then S is a saturated cut and f is maximum.

Proof

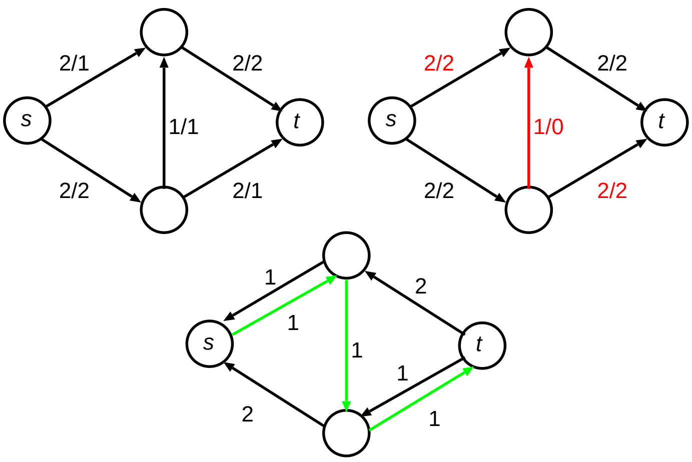
If t is reachable from s in G_f, then f is not maximal.

- Let P be a (simple) path from s to t in G_f.
- Let δ be the minimum residual capacity of an edge in P.
  By definition, $r(e) > 0$, for all edges e in G_f. Therefore, $\delta > 0$.
- Construct a flow f' of value $\mathrm{val}(f) + \delta$:

$$f'(e) = \begin{cases} f(e) + \delta, & \text{if } e' \in P \\ f(e) - \delta, & \text{if } e'' \in P \\ f(e), & \text{if neither } e' \text{ nor } e'' \text{ belongs to } P. \end{cases}$$

- f' is a flow and $\mathrm{val}(f') = \mathrm{val}(f) + \delta$.

Example



If t is not reachable from s in G_f, then f is maximal.

- Let S be the set of nodes reachable from s in G_f, and let $T = V \setminus S$.
- There is no edge (v, w) in G_f with $v \in S$ and $w \in T$.
- Hence
  - $f(e) = \mathrm{cap}(e)$, for any $e \in E \cap (S \times T)$, and
  - $f(e) = 0$, for any $e \in E \cap (T \times S)$.
- Thus S is saturated and, by the Lemma, f is maximal.

Ford-Fulkerson Algorithm

1. Start with the zero flow, i.e., $f(e) = 0$, for all $e \in E$.
2. Construct the residual network G_f.
3. Check whether t is reachable from s in G_f.

- if not, stop.
- if yes, increase the flow along an *augmenting path*, and iterate.

Analysis

- Let $|V| = n$ and $|E| = m$.
- Each iteration takes time $O(n+m)$.
- If capacities are arbitrary reals, the algorithm may run forever.

Integer capacities

- Suppose capacities are integers, bounded by C.
- $v^* \stackrel{\text{def}}{=}$ value of maximum flow $\leq Cn$.
- All flows constructed are integral (proof by induction).
- Every augmentation increases flow value by at least 1.
- Running time $O((n+m)v^*) \rightsquigarrow$ *pseudo-polynomial algorithm*

Edmonds-Karp Algorithm

- Compute a *shortest* augmenting path, i.e. with a minimum number of arcs.
- Apply breadth-first search (or Dijkstra's algorithm).
- Number of iterations is bound by nm, leads to an $O(nm^2)$ maximum flow algorithm.
- Works also for irrational capacities.

Max-Flow Min-Cut Theorem

Theorem
For a network (V, E, s, t) with capacities cap : $E \to \mathbb{R}_+$ the maximum value of a flow is equal to the minimum capacity of an (s, t)-cut:

$$\max\{\text{val}(f) \mid f \text{ is a flow}\} = \min\{\text{cap}(S, T) \mid (S, T) \text{ is an } (s, t)\text{-cut}\}$$

Corollary
For integer capacities cap : $E \to \mathbb{Z}_+$, there exists an integer-valued maximum flow $f : E \to \mathbb{Z}_+$.

III. Matching

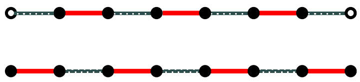
- $G = (V, E)$ undirected graph
- *Matching:* Subset of edges $M \subseteq E$, no two of which share an endpoint.
- *Maximum matching:* Matching of maximum cardinality
- *Perfect matching:* Every vertex in V is matched.

Augmenting paths

- Let M be a matching in $G = (V, E)$.
- A path $P = (v_0, v_1, \dots, v_t)$ in G is called M*-augmenting* if:
  - t is odd,
  - $v_1 v_2, v_3 v_4, v_{t-2} v_{t-1} \in M$,
  - $v_0, v_t \notin \bigcup M = \bigcup_{e \in M} e$.
- If P is an M-augmenting path and $E(P)$ the edge set of P, then

$$M' = M \triangle E(P) = (M \setminus E(P)) \cup (E(P) \setminus M)$$

  is a matching in G of size $|M'| = |M| + 1$.



Berge's Theorem

Theorem (Berge'57)
Let M be a matching in the graph $G = (V, E)$. Then either M is a maximum cardinality matching or there exists an M-augmenting path.

Generic Matching Algorithm

Initialization: $M \leftarrow \emptyset$
Iteration: If there exists an M-augmenting path P, replace $M \leftarrow M \triangle E(P)$.

$\rightsquigarrow$ how can one find an M-augmenting path?

- Difficult in general $\rightsquigarrow$ Edmonds' matching algorithm (Edmonds'65)
- Easy for bipartite graphs

Bipartite graphs

A graph $G = (V, E)$ is *bipartite* if there exist $A, B \subseteq V$ with $A \cup B = V, A \cap B = \emptyset$ and each edge in E has one end in A and one end in B.

Proposition
A graph $G = (V, E)$ is bipartite if and only if each circuit of G has even length.

Bipartite matching

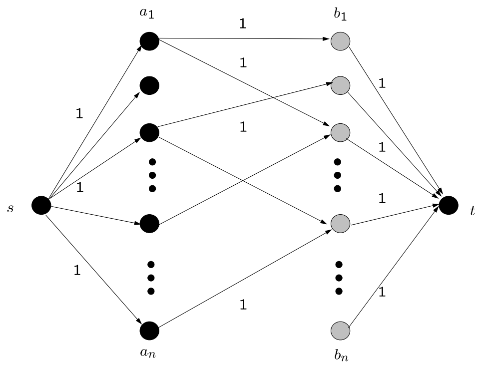
Matching augmenting algorithm for bipartite graphs

Input: Bipartite graph $G = (A \cup B, E)$ with matching M.
Output: Matching M' with $|M'| > |M|$ or proof that no such matching exists.
Description: Construct a directed graph D_M with the same node set as G.
For each edge $e = \{a, b\}$ in G with $a \in A, b \in B$:
if $e \in M$, there is the arc (b, a) in D_M.
if $e \notin M$, there is the arc (a, b) in D_M.
Let $A_M = A \setminus \bigcup M$ and $B_M = B \setminus \bigcup M$.
M-augmenting paths in G correspond to directed paths in D_M starting in A_M and ending in B_M.

Theorem
A maximum-cardinality matching in a bipartite graph $G = (V, E)$ can be found in time $O(|V||E|)$.

Bipartite matching as a maximum flow problem

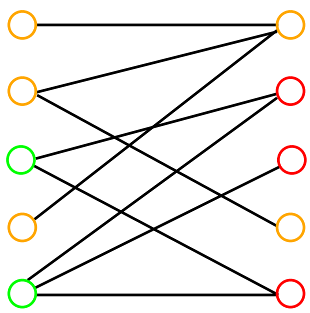
- Add a source s and edges (s, a) for $a \in A$, with capacity 1.
- Add a sink t and edges (b, t) for $b \in B$, with capacity 1.
- Direct edges in G from A to B, with capacity 1.



- Integral flows f correspond to matchings M, with val$(f) = |M|$.
- Ford-Fulkerson takes time $O(nm)$, since $v^* \leq n$.
- Can be improved to $O(\sqrt{n}\,m)$.

Marriage theorem

Theorem (Hall)
A bipartite graph $G = (A \cup B, E)$, with $|A| = |B| = n$, has a perfect matching if and only if for all $B' \subseteq B$, $|B'| \leq |N(B')|$, where $N(B')$ is the set of all neighbors of nodes in B'.

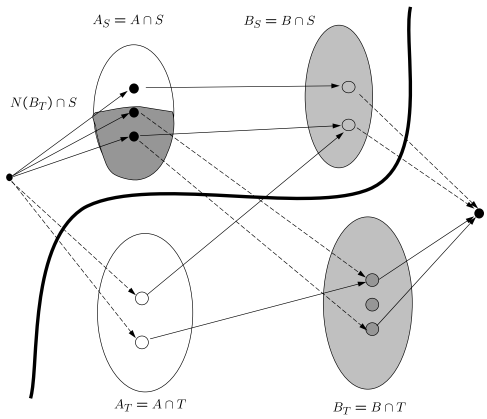


Proof

- Let (S,T) be an (s,t)-cut in the corresponding network.
- Let $A_S = A \cap S, A_T = A \cap T, B_S = B \cap S, B_T = B \cap T$.

$$
\begin{aligned}
\mathrm{cap}(S,T) &= \sum_{e \in E \cap S \times T} \mathrm{cap}(e) \\
&= |A_T| + |B_S| + |N(B_T) \cap A_S| \\
&\geq |N(B_T) \cap A_T| + |N(B_T) \cap A_S| + |B_S| \\
&= |N(B_T)| + |B_S| \\
&\geq |B_T| + |B_S| = |B| = n
\end{aligned}
$$

- By the max-flow min-cut theorem, the maximum flow is at least n.



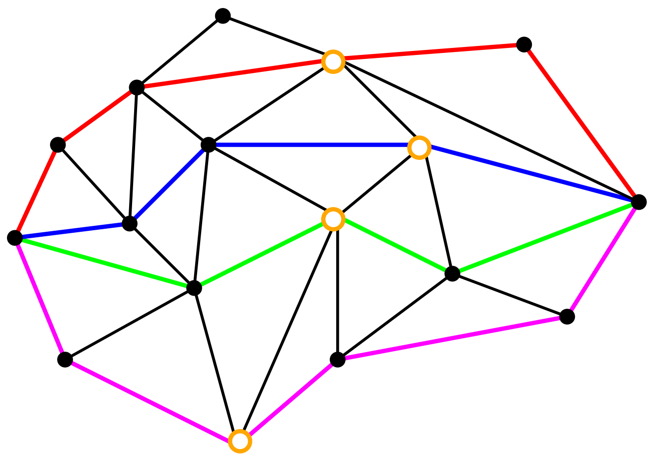
König's theorem

- $G = (V,E)$ undirected graph
- $C \subseteq V$ is a *vertex covering* if every edge of G has at least one end in C.
- **Lemma:** For any matching M and any vertex covering C, we have $|M| \leq |C|$.
- **Theorem (König)** For a bipartite graph G,

$$\max\{|M| : M \text{ a matching }\} = \min\{|C| : C \text{ a vertex covering }\}.$$

Network connectivity

- $G = (V,E)$ directed graph, $s,t \in V, s \neq t$ non-adjacent.
- **Theorem (Menger)** The maximum number of *arc-disjoint* paths from s to t equals the minimum number of arcs whose removal disconnects all paths from s to t.
- **Theorem (Menger)** The maximum number of *node-disjoint* paths from s to t equals the minimum number of nodes (different from s and t) whose removal disconnects all paths from s to t.



Duality in linear programming

- Primal problem

$$z_P = \max\{\mathbf{c}^\mathbf{T} x \mid Ax \leq b, x \in \mathbb{R}^n\} \qquad \text{(P)}$$

- Dual problem

$$w_D = \min\{b^T u \mid A^T u = \mathbf{c}, u \geq 0\} \qquad \text{(D)}$$

General form

(P)			(D)		
min	$c^T x$		max	$u^T b$	
w.r.t.	$A_{i*}x \geq b_i$,	$i \in M_1$	w.r.t	$u_i \geq 0$,	$i \in M_1$
	$A_{i*}x \leq b_i$,	$i \in M_2$		$u_i \leq 0$,	$i \in M_2$
	$A_{i*}x = b_i$,	$i \in M_3$		u_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$		$(A_{*j})^T u \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$		$(A_{*j})^T u \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$		$(A_{*j})^T u = c_j$,	$j \in N_3$

Duality theorems

- **Weak duality** If x^* is primal and u^* is dual feasible, then

$$c^T x^* \leq z_P \leq w_D \leq b^T u^*.$$

- **Strong duality** If both (P) and (D) have a finite optimum, then $z_P = w_D$.
- *Only four possibilities*
  1. z_P and w_D are both finite and equal.
  2. $z_P = +\infty$ and (D) is infeasible.
  3. $w_D = -\infty$ and (P) is infeasible.
  4. (P) and (D) are both infeasible.

Maximum flow and duality

- Primal problem

$$\begin{aligned} \max \quad & \sum_{e:\text{source}(e)=s} x_e - \sum_{e:\text{target}(e)=s} x_e \\ \text{s.t.} \quad & \sum_{e:\text{target}(e)=v} x_e - \sum_{e:\text{source}(e)=v} x_e = 0, \quad \forall v \in V \setminus \{s,t\} \\ & 0 \le x_e \le c_e, \qquad \forall e \in E \end{aligned}$$

- Dual problem

$$\begin{aligned} \min \quad & \sum_{e\in E} c_e y_e \\ \text{s.t.} \quad & z_w - z_v + y_e \ge 0, \quad \forall e = (v,w) \in E \\ & z_s = 1, z_t = 0 \\ & y_e \ge 0, \qquad \forall e \in E \end{aligned}$$

Maximum flow and duality (2)

- Let (y^*, z^*) be an optimal solution of the dual.
- Define $S = \{v \in V \mid z_v^* > 0\}$ and $T = V \setminus S$.
- (S,T) is a minimum cut.
- Max-flow min-cut theorem is a special case of linear programming duality.

Total unimodularity

- A matrix A is *totally unimodular* if each subdeterminant of A is $0, +1$ or -1.
- **Theorem (Hoffman and Kruskal)** $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff the polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b, x \ge 0\}$ is integral, i.e., $P = \text{conv}(P \cap \mathbb{Z}^n)$, for any $b \in \mathbb{Z}^m$.
- **Corollary** $A \in \mathbb{Z}^{m\times n}$ is totally unimodular iff for any $b \in \mathbb{Z}^m, c \in \mathbb{Z}^n$ both optima in the LP duality equation

$$\max\{c^T x \mid Ax \le b, x \ge 0\} = \{\min b^T u \mid A^T u \ge c, u \ge 0\}$$

  are attained by integral vectors (if they are finite).

- **Proposition** The constraint matrix A arising in a maximum flow problem is totally unimodular.

Matching and linear programming

- $G = (V,E)$ undirected graph, $M \subseteq E$ matching
- Incidence vector: $\chi^M : E \to \mathbb{R}$, $\chi^M(e) = \begin{cases} 1, & \text{if } e \in M, \\ 0, & \text{if } e \notin M. \end{cases}$
- Maximum matching as an *integer linear program*

$$\max\{\sum_{e\in E} x_e \mid \sum_{e\ni v} x_e \le 1, \forall v \in V, \;\; x_e \in \{0,1\}, \forall e \in E\}$$

- For **bipartite graphs** the constraint matrix is totally unimodular $\rightsquigarrow$ *linear program*

$$\max\{\sum_{e\in E} x_e \mid \sum_{e\ni v} x_e \leq 1, \forall v \in V,\ \ x_e \geq 0, \forall e \in E\}$$

- *Dual linear program*

$$\min\{\sum_{v\in V} y_v \mid y_v + y_w \geq 1, \forall e = \{v,w\} \in E,\ \ y_v \geq 0, \forall v \in V\}$$

$\rightsquigarrow$ minimum vertex cover

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