Cut-free hypersequent calculus for an essence **S5** logic

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Abstract

In this report, we present a cut-free hypersequent calculus for an essence logic based on **S5** by changing the rules in Restall's cut-free hypersequent calculus for **S5**.

Essence logics have started their history from Fine's papers [3, 4, 5]. These logics deal with the sentences "ϕ is essentially true", i.e. "if ϕ is true, then it is necessarily true". Although cut-free sequent and hypersequent calculi are developed for many modal logics (see, e.g., [9]), it seems that essence logics are still a blind-spot. In our report, we will try to make the first step toward the development of (hyper)sequent calculi for essence logics. We choose an **S5**-style essence logic, since **S5** has at least eight various hypersequent calculi (see, e.g., [1, 6]). Thus, we have a good choice of calculi which can be used as a basis of our own one. We choose Restall's one [8], since it is one of the simplest.

We define two modal languages, $\mathscr{L}_\Box$ and $\mathscr{L}_\circ$, with the alphabets $\langle \mathscr{P}, \neg, \wedge, \rightarrow, \Box, (,) \rangle$ and $\langle \mathscr{P}, \neg, \wedge, \rightarrow, \circ, (,) \rangle$, respectively, where $\mathscr{P} = \{p, q, r, p_1, \ldots\}$ is the set of propositional variables, $\Box$ is a necessity operator, and $\circ$ is an essence operator. We put $\top := p \rightarrow p$. The notion of a formula in these languages is defined in a standard way. We introduce a translation function τ from $\mathscr{L}_\circ$ to $\mathscr{L}_\Box$ as follows: $\tau(\circ\phi) = \tau(\phi) \rightarrow \Box\tau(\phi)$ (the propositional case remains unchanged). Then we can put $\mathbf{S5}^\circ = \{\phi \in \mathscr{L}_\circ \mid \tau(\phi) \in \mathbf{S5}\}$.

Jie Fan [2] formalized $\mathbf{S5}^\circ$ via Hilbert-style calculus which is as follows:

- axioms of **CPL**
- $\circ\top$
- $\neg p \rightarrow \circ p$
- $(\circ p \wedge \circ q) \rightarrow \circ(p \wedge q)$
- $p \rightarrow \circ(\circ\neg p \rightarrow p)$
- $\neg\circ\neg p \rightarrow \circ(\circ\neg p \rightarrow p)$

Inference rules are as follows: (MP) $\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}$, (Sub) $\dfrac{\phi}{\phi[\psi/p]}$, and (R) $\dfrac{\phi \rightarrow \psi}{(\phi \wedge \circ\phi) \rightarrow \psi}$, where $\phi[\psi/p]$ the result of a replacement of all the occurrences of a variable p in ϕ with ψ.

Our cut-free hypersequent calculus for $\mathbf{S5}^\circ$ is obtained from Restall's [8] hypersequent calculus for **S5** by the replacement of the rules for $\Box$ with the rules for $\circ$. The axiom and the rules of our calculus are presented below (the notion of a sequent is understood in a standard way, a hypersequent is a multiset of sequents, the letters $\Gamma, \Delta, \Pi, \Sigma$ stand for finite sets of $\mathscr{L}_\circ$-formulas, and the letters H, G stand for hypersequents).

$$\text{(Ax) } \phi \Rightarrow \phi$$

$$\text{(Merge) } \frac{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H} \qquad \text{(Cut) } \frac{\Gamma \Rightarrow \Delta, \phi \mid H \qquad \phi, \Pi \Rightarrow \Sigma \mid G}{\Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H \mid G}$$

$$\text{(EW}\Rightarrow\text{) } \frac{H}{\phi \Rightarrow \mid H} \qquad (\Rightarrow\text{EW}) \frac{H}{\Rightarrow \phi \mid H} \qquad \text{(IW}\Rightarrow\text{) } \frac{\Gamma \Rightarrow \Delta \mid H}{\phi, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\text{IW}) \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \phi \mid H}$$

$$(\neg\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\phi\mid H}{\neg\phi,\Gamma\Rightarrow\Delta\mid H} \qquad (\Rightarrow\neg)\ \frac{\phi,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta,\neg\phi\mid H}$$

$$(\wedge\Rightarrow)\ \frac{\phi,\psi,\Gamma\Rightarrow\Delta\mid H}{\phi\wedge\psi,\Gamma\Rightarrow\Delta\mid H} \qquad (\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow\Delta,\phi\mid H \qquad \Gamma\Rightarrow\Delta,\psi\mid G}{\Gamma\Rightarrow\Delta,\phi\wedge\psi\mid H\mid G}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\phi\mid H \qquad \psi,\Pi\Rightarrow\Sigma\mid G}{\phi\rightarrow\psi,\Gamma,\Pi\Rightarrow\Delta,\Sigma\mid H\mid G} \qquad (\Rightarrow\rightarrow)\ \frac{\phi,\Gamma\Rightarrow\Delta,\psi\mid H}{\Gamma\Rightarrow\Delta,\phi\rightarrow\psi\mid H}$$

$$(\circ\Rightarrow)\ \frac{\phi,\Gamma\Rightarrow\Delta\mid H \qquad \Pi\Rightarrow\Sigma,\phi\mid G}{\circ\phi,\Pi\Rightarrow\Sigma\mid\Gamma\Rightarrow\Delta\mid H\mid G} \qquad (\Rightarrow\circ)\ \frac{\Rightarrow\phi\mid\phi,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta,\circ\phi\mid H}$$

We have proved the following theorems.

Theorem 1. *For any $\mathscr{L}_\circ$-formula ϕ, it holds that ϕ is provable in Fan's Hilbert-style calculus for* $\mathbf{S5}^\circ$ *iff it is is provable in the hypersequent calculus for* $\mathbf{S5}^\circ$*.*

Theorem 2. *The rule* (Cut) *is admissible in the hypersequent calculus for* $\mathbf{S5}^\circ$*.*

The dual modality for $\Box$ is $\Diamond$ (a possibility operator). It is natural to ask what is the dual modality for $\circ$? The answer is that it is an accident modality $\bullet$. A formula $\bullet\phi$ means that ϕ is accidentally true and is equivalent to the formulas $\phi\wedge\neg\Box\phi$ and $\neg\circ\phi$. At that $\circ\phi=\neg\bullet\phi$. The accident operator was introduced by Small [7] in the context of the investigation of Gödel's ontological argument. We are able to present the hypersequent rules for the **S5**-style accident modality.

$$(\bullet\Rightarrow)\ \frac{\Rightarrow\phi\mid\phi,\Gamma\Rightarrow\Delta\mid H}{\bullet\phi,\Gamma\Rightarrow\Delta\mid H} \qquad (\Rightarrow\bullet)\ \frac{\phi,\Gamma\Rightarrow\Delta\mid H \qquad \Pi\Rightarrow\Sigma,\phi\mid G}{\Pi\Rightarrow\Sigma,\bullet\phi\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

Acknowledgments. The report is supported by the grant from the National Science Centre, Poland, grant number DEC-2017/25/B/HS1/01268.

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