Quantifiers: Higher-order Predicates or Choice Functions?

The interpretation of quantifiers as choice functions has been discussed since the seminal paper of Goldfarb [1979], with the underlying ideas tracing back to the works of David Hilbert, Thoralf Skolem and Jacques Herbrand. The main idea is that the truth of an existential sentence such as $\exists x \varphi x$ allows us to pick an x such that φ. This idea is formalized by the ε-operator first introduced by Hilbert and Bernays [1934]. The ε-operator is a variable-binding operator that forms terms from open sentences, like $\varepsilon x \varphi(x)$, which is interpreted as "an x such that φ, if any". Moreover, the Epsilon Calculus has been proved to be a conservative extension of First-Order Logic [Leisenring 1969]. The extensional semantics of the ε-operator is defined by a choice function which picks out a representative object from each set. However, the representative object is arbitrarily chosen and so the denotation of the ε-operator is indeterminate. Finally, based on the ε-operator, both the universal $\forall$ and existential $\exists$ quantifiers can be defined [Leisenring 1969].

This interpretation challenges the standard one of quantifiers as higher-order predicates, adopted by the Theory of Generalized Quantifiers first developed by Lindstrom [1966]. The idea – which can be traced back to Gottlob Frege – is that, while an existential sentence $\exists x \varphi(x)$ asserts that the set of individuals satisfying the property φ – denoted by A – is not empty, a universal sentence $\forall x \varphi(x)$ states that A is the entire universe of discourse. Moreover, Sher [2012] explains how the truth conditions of quantified First-Order formulas are defined according to the interpretation of quantifiers as higher-order predicates. That is why I will assume that the interpretation of quantifiers as higher-order predicates is adopted by the model-theoretic semantics of First-Order Logic. The two opposite interpretations of quantifiers raise a compelling question: are quantifiers higher-order predicates or choice functions?

In this talk, I will argue for the latter interpretation by pointing out that the Epsilon Calculus best represents the dependence relations among quantified variables – which are instead ruled out by the standard interpretation of First-Order quantifiers. More precisely, I will first argue that the representation of dependence relation among quantified variables is a *desideratum* for the correct interpretation of quantifiers. Following the work of Henkin [1961], I will argue that the semantic notion of dependence among quantified variables follows straightforwardly from the syntactic notions of scope and nested quantifiers. In this sense, the notion of dependence is conservative over the model-theoretic semantics for quantifiers. Moreover, I will characterize this dependence relation as *functional:* the value of a variable is completely determined by that of another – and therefore it can be represented by Skolem Functions [Meyer-Viol 1995]. Finally, I will propose a few examples of dependence relations in the axioms of mathematical and scientific theories formulated in First-Order Logic.

Then, I will adopt the *desideratum* of dependence relations in order to compare and evaluate the two interpretations of quantifiers. I will argue that the comparison relies on the challenge between the metalanguage of First-Order Logic – namely ZFC_{FOL}– and the one of the Epsilon Calculus – namely ZF_{EC}. On the one hand, I will point out that First-Order Logic cannot account for dependence relations, by stressing that First-Order formulas and their Skolem Normal Form are not logically equivalent. I will explain this well-known metalogical result according to ZFC_{FOL}, which is also adopted for the interpretation of quantifiers as higher-order predicates. Given that the representation of dependence relations has been taken as a *desideratum* for the correct interpretation of quantifiers, I will argue that the metalogical result about First-Order formulas and their Skolem Normal Form undermines the interpretation of quantifiers as higher-order predicates – as hinted also by Hintikka [1998].

On the other hand, following Leisenring [1969] theorem, I will point out that Skolem functions can be conservatively added to the Epsilon Calculus – thus allowing to represent the dependency relations. Indeed, the choice function adopted by the extensional semantics of the ε-operator provides an interpretation for the Skolem functions. This result brings about a comparison between ZFC_{FOL} and ZF_{EC}: I will point out that the ε-operator is equivalent to a stronger version of the Axiom of Choice, namely the Axiom of Global Choice [Leisenring 1969]. In conclusion, I will explain why this equivalence does not affect the argument: *if* the representation of dependence relations is taken as a condition for the correct interpretation of quantifiers, *then* both the Axiom of Choice and the Axiom of Global Choice are required to prove the logical equivalence of Skolem Normal Forms with, respectively, their Second-Order and Epsilon Calculus formulas.

§References

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