An Order-Theoretic Approach to Kripkean Theories of Probability and Truth

In the aftermath of Kripke‘s *Outline of a Theory of Truth*, [3], extensions or adaptations of the theory to modal notions such as necessity have been studied. Recently, also Kripkean theories of probability and truth have gained interest (Campbell-Moore 2015, [1]). The objective of this talk is to introduce an order-theoretic mathematical system which allows for a generalisation of Campbell-Moore's approach in the following directions:

(1) The system will be flexible regarding the particular fixed point considered. While the proof strategy "approximation of fixed points", exploiting transfinite induction, works well for the least fixed point, one is able to give uniform and simple existence proofs for let us say the least and greatest intrinsic fixed point in Strong Kleene in an order-theoretic setting. (Note that Kripke himself also hinted at order-theoretic methods when he discusses the existence of a greatest intrinsic fixed point.)

(2) The system will be flexible regarding the underlying evaluation scheme. In truth theories, at least the evaluation schemes Weak Kleene, Strong Kleene, Supervaluation and Belnap's useful four valued logic coexist because of philosophical and technical differences. Campbell-Moore's system focuses on the Strong Kleene scheme, which is a reasonable choice, but raises the question whether the other schemes could be also used. On the math side, our system delivers an affirmative answer, although several philosophical questions are still work in progress.

Technically, our system should be seen as a generalisation of the approach to Kripkean theories of truth presented in Chapter 2 of the book *The Revision Theory of Truth* by Gupta and Belnap [2]. One characteristic of this approach is that hypotheses (for the truth predicate) are represented as functions $h\colon D \to B$, where D denotes the domain of discourse and B the non-classical set of truth values. With Strong Kleene, one would have e.g. $B = \{t_3, f_3, n_3\}$. Note that one assumes that there exists a coding function c coordinating D and the sentences of the first-order language. Those hypotheses are then updated by means of a jump operator $\rho_M\colon (D \to B) \to (D \to B)$. In the case of truth, this operator reads

$$\rho_M(g)(v) = \begin{cases} \mathrm{Val}_{S,(D,C_w,F_w,R_w[G:=g])}(c^{-1}v) & \text{if } v \in c(L_{sen}^{(\mathcal{F},\mathcal{C},\mathcal{R})}), \\ \mathbf{f} & \text{otherwise,} \end{cases}$$

where $M = (D, C_w, F_w, R_w)$ denotes a classical model expect on the relation symbol G standing for the truth predicate, and S the evaluation scheme.[1] This jump operator can be shown to be monotonous for the salient evaluation schemes. As the order-theoretic structure on B—ccpo for $\{t_3, f_3, n_3\}$ and complete lattice in the four-valued case—is inherited to the set of functions mapping to B, as with our set of hypotheses, we end up searching for fixed points of a monotonous operator on a ccpo or complete lattice. Fortunately, Knaster-Tarski

[1] Note that we are dealing with sentences only and hence suppress an index for the assignment.

(or the corresponding theorem about ccpos) can be applied here to prove the existence of the least or greatest (intrinsic) fixed point.

In the setting of probability and truth, we relativise our models and hypotheses to a set of possible worlds W and assume a probability measure $m(w)$ on $\mathcal{P}(W)$ for all worlds $w \in W$. (For examples and motivation of this see [1].) For a combined theory of probability and truth, we take a truth hypothesis $g\colon W \to D \to B$ together with a probability hypothesis $h\colon W \to D \to D \to B$, where the intended interpretation of $h(c_1(\phi), c_2(q))$ should be "The probability of ϕ is greater or equal to q", for a sentence ϕ, a real number q, and appropriate codings c_1, c_2. The jump operator in this setting will operate on tuples and can be represented as a tuple of jump operators for truth and probability respectively, $\rho^{tp}_{\mathfrak{M}} = (\rho_1, \rho_2)$, where the former is just a straightforward variant of the operator shown above.[2] For the latter operator, we would suggest for Strong Kleene—in line with [1]

$$\rho_2 = \lambda w.\lambda v_1.\lambda v_2. \begin{cases} \mathrm{pVal}_S(c_1^{-1}(v_1), c_2^{-1}(v_2)) & \text{if } v_1 \in c_1(L^{(\mathcal{F},\mathcal{C},\mathcal{R})}_{sen}) \text{ and } v_2 \in c_2(\mathbb{Q}), \\ \mathbf{f} \quad \text{otherwise.} \end{cases}$$

$$\mathrm{pVal}_S(\phi, q) = \begin{cases} t_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}_{S,M^*(w_1)}(\phi) = t_3\} \geq q, \\ f_3 & \text{if} \quad m(w)\{w_1 \in W \mid \mathrm{Val}_{S,M^*(w_1)}(\phi) = f_3\} > 1 - q, \\ n_3 & \text{else,} \end{cases}$$

$$\text{where} \quad M^*(w_1) := (D, \mathfrak{C}_w(w_1), \mathfrak{F}_w(w_1), \mathfrak{R}_w(w_1)[G := g(w_1)][H := h(w_1)]).$$

Now, it can easily be shown that the product of two ccpos (complete lattices) is also one. Hence, we have sufficient mathematical structure to transfer the existence proofs from the truth case to this setting.

Unlike [1], we refrain from "closing off", but still are able to prove the most important desiderata for our fixed point theory, such as delivery of P-predicates, well-posedness of lower-upper probabilities, consistency and the Gaifman condition. Concerning consistency in Strong Kleene, we think there is actually an easy proof by contradiction and the transfinite induction suggested in [1, Fn. 17] is not necessary in our framework.

References

[1] Catrin Campbell-Moore. How to express self-referential probability. A Kripkean proposal. *The Review of Symbolic Logic*, 8(4):680–704, 2015.

[2] Anil Gupta and Nuel Belnap. *The Revision Theory of Truth.* MIT Press, 1993.

[3] Saul Kripke. Outline of a theory of truth. *The Journal of Philosophy*, 72(19):690–716, 1975.

[2] We are using the convention that an "old German" letter stands for the respective object relativised to possible worlds, so e.g. $\mathfrak{M}(w_1)$ will be a model such as M for each world w_1.