A Walk through the Landscape of Intuitionistic Modal Logic

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In this talk I want to explore the diversity of the field of intuitionistic modal logics. This area in logic is interesting for philosophers, mathematicians and computer scientists. Since there is no 'one true' meaning of *intuitionistic* modal logic, intuitionistic modal logics can be studied from different angles. I want to investigate what motivates researchers to study these logics. My interest in intuitionistic modal logic started with the study of intuitionistic versions of Gödel-Löb logic GL, see [6]. Classically, GL is the provability logic of Peano Arithmetic where $\Box$ is interpreted as provability. It is interesting to think about provability in an intuitionistic setting. Until now, it is an open question what the provability logic of Heyting Arithmetic is.

My study is a small island in the field of intuitionistic modal logic. Although the landscape is not very large, it is spread over different disciplines and intuitionistic modal logics are often hidden in papers. Therefore, I do not claim to present a complete list. De Paiva, Goré, Mendler [7] and Stewart, de Paiva, Alechina [9] already gave a retrospective on the occasion of the series of workshops on Intuitionistic Modal Logic including a list of literature. They focus on developments in computer science. I want to start from a mathematical point of view and I conclude with some remarks on applications in computer science and philosophy.

In mathematics, it is common to take classical modal logic as a basis for defining intuitionistic related systems, resulting in for example $\mathsf{S4}$-type and $\mathsf{S5}$-type intuitionistic systems. This approach is not particularly motivated by philosophical arguments, but it is rather a technical challenge. The key challenge is how to interpret the modalities, since $\Diamond$ and $\Box$ are in general non-interdefinable. Intuitionistic analogues are studied using different frameworks. In this overview, I want to investigate algebraic semantics, Kripke semantics, natural deduction and sequent calculi for intuitionistic modal logic.

Algebraic semantics are for example used in the work of Bull [2] and Fischer-Servi [5]. They defined algebraic models in terms of Heyting algebras with additional structure. Bull studied an intuitionistic analogue of $\mathsf{S5}$ which can be translated into the one-variable fragment of intuitionistic first-order logic. Fischer-Servi proposed a method to determine the correct intuitionistic analogue of a classical modal logic. Motivated by Gödel's translation of intuitionistic propositional logic into classical $\mathsf{S4}$, she suggested that an intuitionistic modal logic should be determined by a similar translation into a classical bimodal logic. The algebraic work has been extended by Wolters and Zakharyaschev.

Modal intuitionistic Kripke semantics often combines propositional intuitionistic and classical modal Kripke semantics, resulting in models with two accessibility relations. The interaction between the 'intuitionistic' and the 'modal' relation makes it technical challenging. Božić and Došen [1] discuss Kripke models for fragments of analogues of modal logic K. Došen [4] extends the study to frame conditions for modal extensions. Classically equivalent axioms may result in different intuitionistic modal frame conditions.

An important contribution to proof theory is the PhD thesis of Simpson [8]. He gives a natural deduction framework for a large class of intuitionistic modal logics. It is not only a mathematical contribution, but it also contains a philosophical argument how to understand intuitionistic modal logics. It includes six requirements, one of them stating that adding the excluded middle axiom to the logic yields a standard classical modal logic. More recent work in proof theory is done on label-free natural deduction, tree sequents and nested sequents.

In computer science, intuitionistic modal logics are studied using extensions of the Curry-Howard isomorphism connecting proofs to programs. This field is sometimes referred to as constructive modal type theory. Its goal is to apply intuitionistic modal logics in computations, hardware verification, and security. Sometimes, applications in computer science are not considered as applications of intuitionistic modal logic, simply because researchers do not recognize them as such. As a consequence, those results may not be known among mathematical logicians.

The philosophical contribution in the field of intuitionistic modal logic is not as developed as the other approaches to the field discussed above. Because of the plurality of interpretations in intuitionistic modal logic, it is difficult to find a philosophical explanation to justify those logics. However, there are a few works based on philosophy such as Simpson's work. For example, an interesting but not well-known paper is written by DeVidi and Solomon [3]. They argue that in some situations one does not have to give up the interdefinability of $\Box$ and $\Diamond$ to keep an intuitionistic account of the modalities. To conclude, Bull described that if $\mathsf{S5}$ is the logic of 'all possible worlds', then his intuitionistic analogue is the logic of 'all possible intuitionists' [2].

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