A Tableaux system for V-logic

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Infinitary logics were mainly developed during the 70's by Barwise (1975) and Keisler (1971). In particular, V-logic, first introduced in Barwise (1969) as $\mathfrak{M}$-*logic*, is particularly useful for investigating matters in set theory. The main reason behind this is that such logic allows one to code (and interpret) extensions[1] of V and code proofs as well-founded trees (see Antos, Barton, and S.-D. Friedman (2017).). For this reason, it was chosen by Friedman to develop the so-called Hyperuniverse – a multiverse of set-theory based on V-logic (Arrigoni and S. Friedman, 2013).

Let κ be an inaccessible cardinal. The infinitary language $\mathcal{L}_{\kappa,\omega}$ is defined from the usual first order language, but with infinitary conjunctions ($\bigwedge \Phi$, where Φ is a set of formulas) and disjunctions ($\bigvee \Phi$) of length up to κ. As usual, only a finite number (i.e. less than ω) of quantifiers is admitted in front of formulas. To this language we add (i) a new unary relation symbol $\bar{V}$, representing the ground universe and (ii) a new constant $\bar{a}$ for every $a \in V$; (iii) new constants $\bar{W}_0, \ldots, \bar{W}_\kappa$, for the extensions of that universe. We also add the membership relation $\in$, and call this new language $\mathcal{L}_V$.

The choice of κ is arbitrary; thus the framework allows for the introduction of a very strong infinitary language $\mathcal{L}_{\kappa,\omega}$, where κ is e.g. a Mahlo, or even a measurable, cardinal. The only relevant difference is that while some such languages are complete, others are not (for details, see Dickmann (1975)). Moreover, if we interpret the new constant $\dot{V}$ in set theoretic terms, i.e. as a ground universe of forcing extensions, then that large cardinal also defines how "high" the universe is.[2]

V-logic is the logic based on the language $\mathcal{L}_V$ with the following provability relation $\vdash_V$: (i) modus ponens; (ii) the *Set Rule*: $\{\varphi(\bar{a}) | a \in V\} \vdash_V \forall x, \varphi(x)$; (iii) the V*-rule*: $\{\varphi(\bar{b}) | b \in A\} \vdash \forall x \in \bar{a}, \varphi(x)$. Moreover, V-logic consists of the following axioms: (i) the usual axioms of first order logic; (ii) $\bar{x} \in \bar{V}$ for every $x \in V$; (iii) every atomic or negated atomic sentence of $\mathcal{L}_V \cup \{\bar{x} | x \in V\}$ true in V is an axiom of V-logic.

In this paper, I introduce a tableaux system for such a logic. A new tableaux system clarifies how this kind of logic works, since the connection between syntax and semantics is more explicit. On top of the known advantages of using a tableaux system, such a change enables us to better investigate the problem of completeness for this kind of logics, and ultimately will help us developing set theoretic multiverses based on them.

To build a model of V-logic we will first need to define some crucial properties for this logic, namely its consistency properties. A consistency property is a set S of countable sets of sentences with certain properties. This definition was first introduced by Keisler (1971) and later revised by Barwise (1975), and is needed to prove the Model Existence Theorem. For example, the simplest possible consistency property for V-logic would be the set of all countable sets s of sentences of $\mathcal{L}_V$ such that s has a model A whose domain is the set of new constants added to the language. Following Barwise, to these properties we also add some rules about equality (since our goal is to apply V-logic to set theory).

I now introduce a tableaux system for V-logic as follows. First of all, to the usual rules for connectives and quantifiers we add the following infinitary linear rules:

- $\bigwedge \Phi \vdash_V \varphi_0, \ldots, \varphi_n$, for every $\varphi \in \Phi$
- $\neg \bigvee \Phi \vdash_V \neg\varphi_0, \ldots, \varphi_n$, for every $\varphi \in \Phi$

and branching rules:

[1] Here and throughout the paper with extensions I mean all the possible forcing extensions and widening of the ground universe.

[2] Although this wasn't Barwise's interpretation, this is how V-logic is currently interpreted, e.g. by Friedman.

- $\neg \bigwedge \Phi \vdash_V \neg\varphi_0|\neg\varphi_1|\ldots|\neg\varphi_n$, for every $\varphi \in \Phi$;
- $\bigvee \Phi \vdash_V \varphi_0|\varphi_1|\ldots|\varphi_n$, for every $\varphi \in \Phi$.

A branch of the tableaux tree is *closed* if and only if both a formula φ and its negation $\neg\varphi$ appears in it, and the whole tableaux tree is closed if and only if all its branches are closed.[3] To this usual set up of a tableaux system, we add the consistency properties introduced by Barwise (1975), but arranged in tree-membership properties. As an example, without going in too lengthy details, consider the first of the consistency properties, the *Triviality Rule*. This rules states that $0 \in S$, and that if $s \subseteq s' \in S$, then $s \cup \{\varphi\} \in S$ for each $\varphi \in s'$, where S is a set of sets of sentences of $\mathcal{L}_V$. To turn this into a tree definition for a tableaux system, we define S as the proof tree and each s as its branches. Everything else follows as in Barwise's system.

In this tableuax system all the results proved for the original V-logic, the Model Existence Theorem, V-Completeness Theorem and the Omitting Type Theorem, can also be proved.

A tableaux system is the best kind of proof system for this kind of logic, since it perfectly represent the connection between syntax and semantics. The main reason behind this is the fact that every tableaux style syntactic proof can also be regarded as producing a model. Thus, when proving a particular sentence in V-logic, we will also be producing a model extension of V! This a also evident from the following theorem, that applies determinacy techniques to our context:

Theorem 1. *Let $\Gamma \vdash_V \varphi$ be a proof of φ from Γ in V-logic. This proof is also an open game. Then its determinacy is absolute between models of ZFC.*[4]

A game is open if and only if its outcome set (if we think about the game as a tree, then all its leaves) is open (i.e. it does not contain its boundary points), and it is determined if and only if there is a winning strategy for one of the players. What this means in our context is that given a proof in V-logic, there is always a path in the proof to produce a model. Moreover, this fact is absolute in ZFC, i.e. it cannot be changed through forcing. I submit that such a "gamification" of V-logic would be quite useful in the development of a multiverse conception of set theory. In particular, it is a useful tool in the construction of the V-logic multiverse, for example in solving the problem of completeness.

References

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[3] For an in depth discussion of tableaux, see D'Agostino et al. (2013).

[4] Many thanks to Toby Meadows for the idea and discussion about this theorem.