

SEMINAR: ℓ^2 -BETTI NUMBERS

The Euler characteristic of finite CW-complexes is multiplicative under finite-sheeted coverings and it is homotopy invariant. These properties can be deduced from different descriptions:

- (1) As the alternating sum of the numbers of cells, which are multiplicative but not homotopy invariant.
- (2) As the alternating sum of Betti numbers, which are homotopy invariant but not multiplicative. The n -th Betti number of X is the \mathbb{Q} -dimension of the homology $H_n(X; \mathbb{Q})$ with rational coefficients.
- (3) As the alternating sum of L^2 -Betti numbers, which enjoy the best features from both worlds: they are multiplicative and homotopy invariant. The n -th ℓ^2 -Betti number of X is the von Neumann dimension of the homology $H_n(X; \mathcal{R}(\pi_1 X))$ with coefficients in the group von Neumann algebra.

The ℓ^2 -Betti numbers are meaningful topological invariants, as they obstruct the structures of mapping tori and S^1 -actions. They also have applications to group theory by considering the ℓ^2 -Betti numbers of classifying spaces. Moreover, ℓ^2 -Betti numbers are related to famous open problems, such as the Hopf and Singer conjectures on the Euler characteristic of manifolds, and the Kaplansky conjecture on zero divisors in group rings.

References: We will mostly follow the book by Kammeyer [Kam19]. For complementary reading, we recommend the books by Löh [Löh20] and Lück [Lüc02].

Prerequisites: Basic knowledge of topology and group theory is required.

1. CONSTRUCTION AND EXAMPLES

Talk 1 (The group von Neumann algebra). We briefly review some results about Hilbert spaces. We introduce a central object of the seminar: the group von Neumann algebra, which is a completion of the complex group ring.

References: [Kam19, Section 2.1 and 2.2].

Key statements: Theorem 2.12, Theorem 2.19, Definition 2.23, Theorem 2.24, Examples 2.25 and 2.26.

Talk 2 (The von Neumann dimension). We define the von Neumann dimension of Hilbert modules and discuss its properties. This is the algebraic foundation for the theory of ℓ^2 -Betti numbers.

References: [Kam19, Section 2.3].

Key statements: Theorem 2.36, Definition 2.37, Examples 2.38–2.42, Theorem 2.44.

Talk 3 (G -CW-complexes). A G -CW-complex is a CW-complex on which the group G acts. These are the nice spaces in equivariant algebraic topology. The cellular chain complex of a G -CW-complex is a $\mathbb{Z}G$ -chain complex, and its ℓ^2 -completion is a Hilbert chain complex.

References: [Kam19, Sections 3.1 and 3.2].

Key statements: Theorem 3.2, Proposition 3.5, Theorem 3.11.

Talk 4 (ℓ^2 -Betti numbers). We arrive at a milestone of the seminar: the definition of ℓ^2 -Betti numbers. We discuss first examples and tools for computations.

References: [Kam19, Section 3.3].

Key statements: Definition 3.13, Examples 3.14–3.17, Theorem 3.18, Theorem 3.19.

2. APPLICATIONS TO TOPOLOGY, ALGEBRA, AND GROUP THEORY

Talk 5 (Applications to open problems). ℓ^2 -Betti numbers have applications to different areas of mathematics. They are related to famous conjectures of Hopf and Kaplansky. We also briefly discuss cohomological ℓ^2 -Betti numbers and the ℓ^2 -version of Poincaré duality.

References: [Kam19, Chapter 1, Sections 3.4 and 3.5].

Key statements: Corollary 3.20, Theorem 3.25, Example 3.26, Theorem 3.32.

Talk 6 (Applications to topology). ℓ^2 -Betti numbers are meaningful topological obstructions. They vanish in the presence of self-coverings, mapping torus structures, and circle actions.

References: [Kam19, Section 3.6].

Key statements: Proposition 3.34, Theorem 3.38, Theorem 3.39, Theorem 3.40, Corollary 3.41.

Talk 7 (ℓ^2 -Betti numbers of general G -spaces). We extend the definition of ℓ^2 -Betti numbers from proper finite type G -CW-complexes to arbitrary G -spaces. We also introduce classifying spaces for families of subgroups.

References: [Kam19, Sections 4.1, 4.2, and 4.3].

Key statements: Theorem 4.2, Theorem 4.4, Definition 4.6, Definition 4.9, Theorem 4.10.

Talk 8 (Applications to group theory). The ℓ^2 -Betti numbers of a group are defined as the ℓ^2 -Betti numbers of its classifying space. They have applications to co-Hopfian groups, deficiency, and one-relator groups.

References: [Kam19, Sections 4.4 and 4.5].

Key statements: Definition 4.12, Theorem 4.14, Table 4.1, Theorem 4.15, Theorem 4.17, Theorem 4.20, Theorem 4.21, Section 4.5.4.

3. LÜCK'S APPROXIMATION THEOREM

Talk 9 (Lück's approximation theorem: Statement and functional calculus). Lück's approximation theorem is a cornerstone of the theory. It expresses the ℓ^2 -Betti numbers as limits of ordinary Betti numbers. We discuss the statement and preparations for the proof.

References: [Kam19, Sections 5.1 and 5.2].

Key statements: Theorem 5.2, Theorem 5.10, Theorem 5.14, Proposition 5.15.

Talk 10 (Lück's approximation theorem: Proof and extensions). We prove Lück's approximation theorem and discuss some extensions.

References: [Kam19, Sections 5.3 and 5.4].

Key statements: Section 5.3, Theorem 5.19.

Talk 11 (Rank gradient and cost). The first ℓ^2 -Betti number of groups has non-abelian and dynamical counter-parts: the rank gradient and the cost. They are related by inequalities.

References: [Kam19, Section 5.5]. See also [Löh20, Section 4.3].

Key statements: Theorem 5.29, Inequality (5.8), Theorem 5.33, Theorem 5.36 in [Kam19].

Talk 12 (The approximation conjecture). The approximation conjecture is a (conjectured) generalisation of Lück's approximation theorem. It has applications to the Atiyah conjecture.

References: [Kam19, Section 5.6].

Key statements: Conjecture 5.38, Theorem 5.40, Conjecture 5.43, Theorem 5.45, Theorem 5.51, Theorem 5.52.

REFERENCES

- [Kam19] Holger Kammeyer. *Introduction to ℓ^2 -invariants*, volume 2247 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.
- [Löh20] Clara Löh. *Ergodic theoretic methods in group homology—a minicourse on L^2 -Betti numbers in group theory*. SpringerBriefs in Mathematics. Springer, Cham, 2020.
- [Lüc02] Wolfgang Lück. *L^2 -invariants: theory and applications to geometry and K -theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 2002.