

Seminar: Topology of 3-manifolds . Summer 2013

We expect that you have read the following pages and have familiarized yourself with the definitions, concepts and results contained in them. We will briefly summarize these in the first session on April 8. If you have any questions contact one of us, preferably before April 8.

A. Manifolds.

A1. Definition. An n -dimensional manifold (n -mf.) is a Hausdorff space M with countable basis such that every point x has a neighbourhood homeomorphic to an open subset of $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Points having a neighborhood homeomorphic to an open subset of $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : x_n > 0\}$ are called interior points, their union is denoted by $\text{int } M$. The other points are called boundary points, their union is denoted by ∂M .

A.2 Examples: (i) $\mathbb{R}_+^n$; $\mathbb{R}_+^n = \text{int}(\mathbb{R}_+^n)$,

$$\partial \mathbb{R}_+^n = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}_+^n.$$

(ii) $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. n-ball

$\text{Int } B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ open n-ball

$\partial B^n = \{x \in \mathbb{R}^n : \|x\| = 1\} =: S^{n-1}$ (n-1)-sphere

(iii) S^n , $\text{int } S^n = S^n$, $\partial S^n = \emptyset$.

(iv) Surfaces : We only give pictures.



S^2



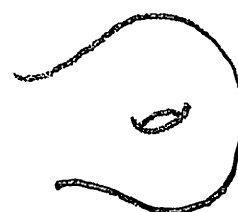
2-torus T^2
 $\cong S^1 \times S^1$



1

2

...



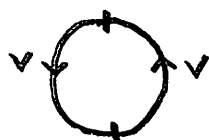
g

closed orientable surface of genus g , Σ_g

Thus $\Sigma_1 = \text{torus}$; we set $\Sigma_0 = S^2$.

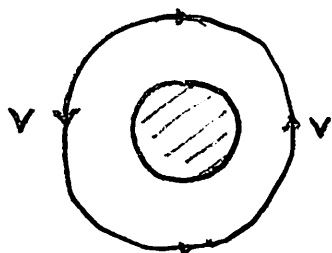
Non-orientable surfaces are hard to draw : it is best to describe them as quotient spaces

$\mathbb{R}P^2$, real projective 2-space.

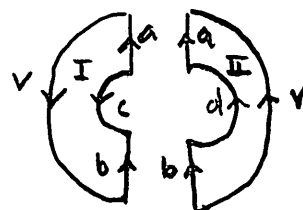


means : identify the 2-edges called v so that the arrows match.

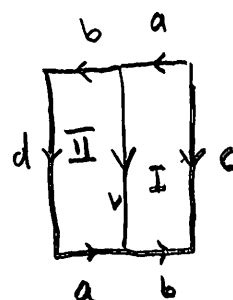
This is a 2-manifold without boundary. To get a feeling for $\mathbb{R}P^2$ remove an open disk around the center. (Now we have a 2-manifold with boundary)



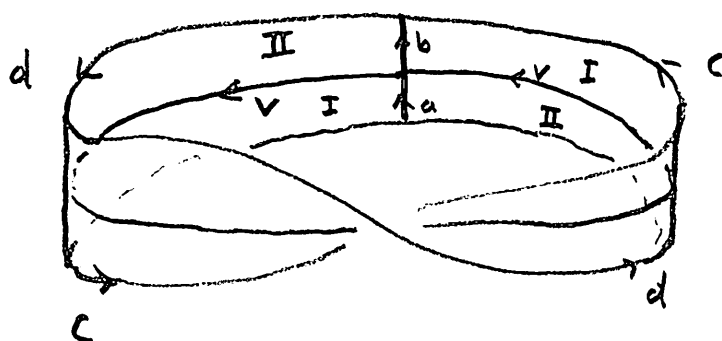
cut into 2 pieces



identify along v as requested



now identify the a 's and the b 's to make the cut undone



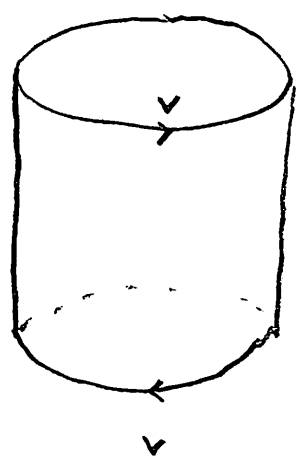
and we get the Möbius strip.

we get $\mathbb{R}P^2$ by glueing the remove disk back along the boundary of the strip. In $\mathbb{R}^3$ this cannot be done without self intersections. But it is easy to do in $\mathbb{R}^4$.

Klein bottle : 1st description:

Glue 2 Möbius strips along their boundary circles

2nd description: identify top and bottom of the cylinder surface, again matching arrows
 $\cong S^1 \times [0, 1]$



The boundary of the cylinder surface is a disjoint union of two 1-spheres

Non-orientable surface of genus $k > 0$.

$k=1$. $\mathbb{R}P^2$

$k > 1$ Let B_1, B_2 be 2 disjoint 2-balls in $\mathbb{R}P^2$ and $\overset{\circ}{B}_1, \overset{\circ}{B}_2$ their interiors

Then $\mathbb{R}P^2 - (\overset{\circ}{B}_1 \cup \overset{\circ}{B}_2)$ is a 2-manifold with boundary $\partial B_1 \sqcup \partial B_2$ ($\mathbb{R}P^2 - (\overset{\circ}{B}_1 \cup \overset{\circ}{B}_2)$ is a Möbius strip with an open disk removed)

Then N_k for $k > 1$ is $\underbrace{\quad}_{k-2 \text{ copies}}$

$$\begin{aligned} & \text{Möbius} \cup (\mathbb{R}P^2 - (\overset{\circ}{B}_1 \cup \overset{\circ}{B}_2)) \cup (\mathbb{R}P^2 - (\overset{\circ}{B}_1 \cup \overset{\circ}{B}_2)) \cup \dots \\ & \quad \partial M = \partial B_1 \quad \quad \quad \partial B_2 = \partial B_1 \\ & \quad \quad \quad \cup \text{ Möbius} \\ & \quad \quad \quad \partial B_2 = \partial M \end{aligned}$$

Here $\bigcup$ means: identify the 1-sphere
 $\partial B_2 = \partial B_1$ ∂B_2 on the left with the 1-sphere
 ∂B_1 on the right, and

similarly for $\bigcup$
 $\partial M = \partial B_1$, $\bigcup$
 $\partial B_2 = \partial M$

Surfaces and their fundamental groups were discussed
 in Topology 1; their homology groups in Topology 2

A.3 Theorem: (a) Up to homeomorphism the only
 connected compact 1-manifolds are S^1 and
 $[0,1]$
 (b) Up to homeomorphism the only connected
 compact 2-manifolds are

$$\Sigma_{g,n} \quad , \quad g = 0, 1, 2, \dots \quad n = 0, 1, 2, \dots$$

$$\text{and} \\ N_{g,n} \quad , \quad g = 1, 2, 3, \dots \quad n = 0, 1, 2, \dots$$

Here $\Sigma_{g,n}$ is Σ_g with the interiors of n
 disjoint 2-balls removed. Similarly $N_{g,n}$.

B. Simplicial complexes.

B.1 Definitions

Points $x_0, \dots, x_k$ in $\mathbb{R}^N$ are called independent if the affine subspace of $\mathbb{R}^N$ spanned by $\{x_0, \dots, x_k\}$ has dimension k .

This is equivalent to :

$D_i := \{x_j - x_i : j \neq i\}$ is linearly independent. for some i
and consequently for all i)

A k -simplex in $\mathbb{R}^N$ is the convex hull of an independent set $\{x_0, \dots, x_k\}$. We write $s(x_0, \dots, x_k)$ for this k -simplex.

$$s(x_0, \dots, x_k) = \left\{ x \in \mathbb{R}^N : \exists \lambda_0, \dots, \lambda_k \in [0, \infty) \text{ with } \sum \lambda_i = 1 \text{ and } x = \sum \lambda_i x_i \right\}$$

The $\lambda_0, \dots, \lambda_k$ are uniquely determined by x and are called the barycentric coordinates of x .

If $\{y_0, \dots, y_\ell\} \subset \{x_0, \dots, x_k\}$ with $y_i \neq y_j$ for $i \neq j$ then $s(y_0, \dots, y_\ell) \subseteq s(x_0, \dots, x_k)$ and is called an ℓ -face of $s(x_0, \dots, x_k)$. We write $s(y_0, \dots, y_\ell) \subset s(x_0, \dots, x_k)$.

The union of $(k-1)$ -faces of $s(x_0, \dots, x_k)$ is called the boundary of $s(x_0, \dots, x_k)$ and is denoted by

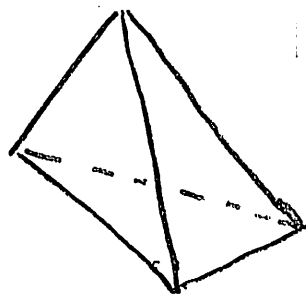
$\dot{s}(x_0, \dots, x_k)$
For a simplex σ with boundary ∂ we call $\sigma^\circ := \sigma - \partial$ an open simplex.

B.1.a Examples: a 0-simplex is point
a 1-simplex $s(x_0, x_1)$ is the set of points between x_0 and x_1 on the line through x_0, x_1

a 2-simplex is a triangle



a 3-simplex is a tetrahedron



Thus a k -simplex is a k -dimensional object. We call k the dimension of $\sigma(x_0, \dots, x_k)$.

B.2 Definition:

A simplicial complex in $\mathbb{R}^N$ is a set K of simplices in $\mathbb{R}^N$ satisfying:

- (sci) For all $x \in \mathbb{R}^N$ there is $\varepsilon > 0$ such that $B_\varepsilon(x) := \{y \in \mathbb{R}^N : \|y - x\| \leq \varepsilon\}$ intersects only finitely many simplices of K
- (scii) $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$
- (sciii) $\sigma, \tau \in K$; then $\sigma \cap \tau$ is a common face of σ and τ

B.2a Examples: Let σ be a k -simplex in $\mathbb{R}^N$.

- (i) Let $K_\sigma := \{\tau : \tau \leq \sigma\}$ is a simplicial complex
- (ii) $K_\sigma := \{\tau : \tau \leq \sigma\}$ is a simplicial complex
- (iii) Let K be a finite simplicial complex in $\mathbb{R}^{N-1}$ and $x \in \mathbb{R}^N, \mathbb{R}^{N-1}$

for $\sigma = s(x_0, \dots, x_k) \in \mathbb{R}^{N-1}$ denote

$s(x, x_0, \dots, x_k)$ by $x * \sigma$.

Define

$$x * K := K \cup \{s(x)\} \cup \{x * \sigma : \sigma \in K\}$$

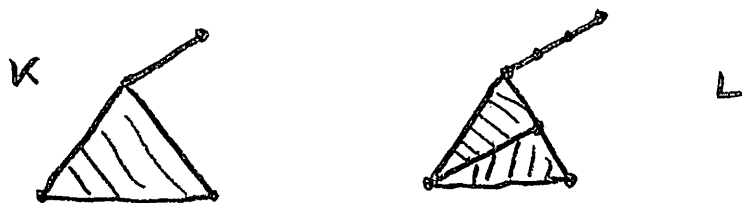
This is a simplicial complex. We often denote $x * K$ by CK , "the cone of K (with cone point x)".

B.3 Def. If K is a simpl. complex in $\mathbb{R}^N$ we call

$$|K| := \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^N \text{ with the subspace topology}$$

the underlying space of K

A subdivision of K is a simplicial complex L with $|L| = |K|$ such that every simplex of L is contained in a simplex of K .



We have encountered in the proof of the excision property for singular homology a standard procedure to subdivide a simplex into simplices of smaller diameter:

Barycentric subdivision: (See next page)

B.4 Example. Barycentric subdivision.

For a simplex $\sigma = s(x_0, \dots, x_k)$ let $\hat{\sigma} := \sum_{i=0}^k \frac{1}{k+1} x_i$ be its barycenter. If σ is a simplex and

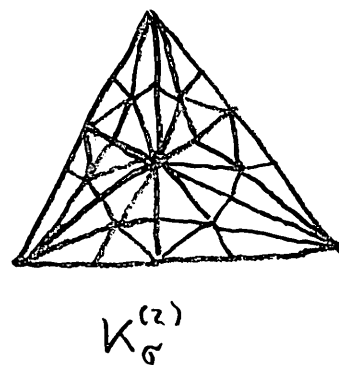
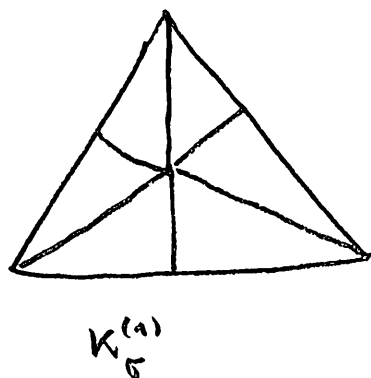
$\tau_0 \subsetneq \tau_1 \subsetneq \dots \subsetneq \tau_r = \sigma$ a strictly increasing sequence of faces, then $\{\hat{\tau}_0, \dots, \hat{\tau}_r\}$ is independent and $s(\hat{\tau}_0, \dots, \hat{\tau}_r) \subset \sigma$ is an r -simplex.

Given a simpl. cx. K in $\mathbb{R}^N$ let

$$K^{(1)} := \left\{ s(\hat{\tau}_0, \dots, \hat{\tau}_r) : \tau_0 \subsetneq \dots \subsetneq \tau_r \text{ is a strictly ascending sequence of faces of } \tau_r \text{ with } \tau_r \in K \right\}$$

Then $K^{(1)}$ is a subdivision of K called the first barycentric (or the first standard) subdivision of K . Iterating this process leads to higher barycentric subdivisions.

$$K_{\sigma}^{(2)}, \quad \sigma = s(x_0, x_1, x_3)$$



Shows that drawings of iterated subdivisions become very soon very complex

B.5 Definition (Star & Link).

Let K be a simplicial complex and $\sigma \in K$

$$(i) \quad St_K(\sigma) := \bigcup_{\sigma < \tau \in K} \tau^{\circ} \quad \text{of open simplices having } \sigma \text{ as a face}$$

is called the open star of σ . It is a subset of $|K|$

$$(ii) \quad \overline{St}_K(\sigma) := \{ \tau \in K : \exists \tau' \in K \text{ with } \sigma < \tau' \text{ and } \tau < \tau' \}$$

is a subcomplex of K , i.e. a subset of K which is itself a simplicial complex.

$$(iii) \quad Lk_K(\sigma) := \{ \tau \in \overline{St}_K(\sigma) \mid \tau \cap \sigma = \emptyset \}$$

B.6. Properties: (i) $\{St_K(v) : v \text{ a 0-simplex of } K\}$ is an open cover of $|K|$.

(ii) $St_K(\sigma)$ and $|\overline{St}_K(\sigma)|$ are contractible.

In fact, for every $x \in \sigma$ and $y \in |\overline{St}_K(\sigma)|$ there is a unique straight segment $[x, y]$ in $|\overline{St}_K(\sigma)|$ from x to y s.t. $[x, y] \cap [x, y'] = \{x\}$ if $y \neq y'$; and straight line means, there is a simplex τ of $\overline{St}_K(\sigma)$ containing x, y and $[x, y]$ is the connecting straight segment in τ .

Thus for $x \in \sigma$ $f_x(y, t) = t \cdot x + (1-t)y$, $y \in |\overline{St}_K(\sigma)|$, $t \in [0, 1]$ defines a contraction.

If $x \in \sigma^{\circ}$, $y \in St_K(\sigma)$, then $[x, y] \in St_K(\sigma)$, so that the same map defines a contraction of $St_K(\sigma)$.

(iii) Let $\{x_0, \dots, x_k, y_0, \dots, y_\ell\}$ be independent. We write $s(x_0, \dots, x_k) * s(y_0, \dots, y_\ell)$ for $s(x_0, \dots, x_k, y_0, \dots, y_\ell)$.

Then $\overline{St}_K(\sigma) = \{\sigma * \tau \mid \tau \in Lk_K(\sigma)\}$.

Maps between simplicial complexes.

Usually one makes a distinction between maps between simplicial complexes and their induced maps between the underlying spaces. We are mostly interested in the latter.

Confusion might occur since $|K| = |L|$ does not imply $K = L$.

B.7 Convention: When we write $|K|$ we mean the underlying space together with the complex K .

When we write $|K| = |L|$ we mean that the ^{only} underlying spaces are the same and not necessarily the complexes.

B.8 Definition. Let K, L be simpl. complexes. (i) A map $f: |K| \rightarrow |L|$ is called simplicial, if the restriction of f to any simplex of K maps this simplex affinely onto a simplex of L .

(ii) $f: |K| \rightarrow |L|$ is called p.l. (piece-wise linear) if there are subdivisions K' of K and L' of L such that $f: |K'| \rightarrow |L'|$ is simplicial.

B.9 Remark. A simplicial map is completely determined once we know what it does on 0-simplices.

Conversely, a map f^0 from the 0-simplices of K into the 0-simplices of L defines a simplicial map iff the image set of a set of 0-simplices of a simplex of K is the set of 0-simplices of a simplex of L .

B.10 Remark. (i) It is easy to show that the composition of simplicial maps is simplicial

(ii) It is a bit of a challenge to show that the composition of p.l. maps is again p.l.

Thus we have two categories:

SC : simplicial complexes and simplicial maps

PL : simplicial complexes and p.l. maps.

We extend these categories a little.

B.11 Definition. A homeomorphism $f: |K| \rightarrow X$ between the underlying space of a simplicial complex and a topological space X is called a triangulation of X .

Two triangulations

$$f: |K| \rightarrow X$$

and

$$g: |L| \rightarrow X$$

are called equivalent if $g^{-1} \circ f: |K| \rightarrow |L|$ is a p.l. map

B.12 Remark: A p.l. map $f: |K| \rightarrow |L|$ which is a homeomorphism is an isomorphism in PL: if K', L' are subdivisions s.t.

$$f: |K'| \rightarrow |L'|$$

is simplicial then f maps every k -simplex by an affine homeomorphism onto a k -simplex of L' . Thus f induces a bijective map $K \xrightarrow{\tilde{f}} L$, s.t.

$$\sigma \in K, \tau < \sigma \text{ then } \tilde{f}(\tau) < \tilde{f}(\sigma).$$

B.13 Def. A triangulated space X is a space with a triangulation $f: |K| \rightarrow X$. A triangulable space is one which admits (sometimes non-equivalent) triangulations.

C. PL- (or combinatorial) manifolds

For several reasons a PL-manifold is not simply a triangulated manifold. One demands further that the triangulation

$f: |K| \rightarrow M^n$ is combinatorial (or p.l.). This is a requirement on K : ($\dim M^n = n$)

(PL-triang.): The link of ^{each} 0 -simplex ^{v} of K is

p.l. homeomorphic to $\begin{cases} \text{the boundary of an } n\text{-simplex, if } f(v) \in \text{int } M \\ \text{an } (n-1)\text{-simplex, if } f(v) \in \partial M \end{cases}$

Fortunately, any triangulation of an n -manifold is a PL-triangulation if $n \leq 4$. This is no longer true for $n \geq 5$.

In dimensions 1, 2, 3 even more is true

C.1 Let $n = 0, 1, 2, 3$. Then any n -manifold admits up to equivalence exactly one triangulation

For $n = 0, 1$ this is trivial, once you know that any conn. 1-manifold is homeomorphic to S^1 , $[0, 1]$, $(0, 1]$ or $(0, 1)$.

For $n = 2$ one must show that any 2-manifold has some triangulation. After this fact, it is not too difficult to prove C.1 for $n = 2$ for compact 2-manifolds.

$n = 3$ is non-trivial.

Since we are only interested in 3-manifolds and their submanifolds we can safely assume:

- (i) All manifolds are triangulable
 (ii) All triangulations are p.l.
 (iii) Any two triangulations are equivalent

C-2 More definitions:

We pass many notions for simplicial complexes to PL-n-manifolds (n arbitrary)

- (i) polyhedron in $|K|$ is $|L|$ where $L \subset K$ is a finite subcomplex

polyhedron in M

$$\begin{array}{ccc} |K| & \xrightarrow{f} & M \\ \uparrow & & \uparrow \\ |L| & \xrightarrow{f|_{|L|}} & f(|L|) \end{array}$$

f a p.l. triangulation

$f(|L|)$ polyhedron in M

- (ii) p.l.-map

$$\begin{array}{ccc} |K| & \xrightarrow{f} & M \\ \alpha \downarrow & & \downarrow \beta \\ |L| & \xrightarrow{g} & N \end{array}$$

f, g p.l. triangulations, α p.l., diagram commutative. Then β is called a p.l. map

- (iii) p.l.-embedding should be clear

β is a p.l. embedding if in the above diagram

α is p.l. and an embedding.

- (iv) p.l. submanifold is the image of a p.l. embedding.

Remark. The notion of p.l. embedding extends in an obvious way to triangulated spaces, i.e. to spaces with an equivalence class of triangulations.

c.3 Def. (Isotopy). Let $f_0, f_1: X \rightarrow Y$ be p.l. embeddings. Then f_0 and f_1 are called isotopic if there exists a level preserving p.l. embedding

$$f: X \times I \longrightarrow Y \times I$$

extending f_0 and f_1 .

Level preserving: $f(x, t) \in Y \times \{t\}$, $0 \leq t \leq 1$.

Extending f_0 and f_1 : $f(x, 0) = (f_0(x), 0)$
 $f(x, 1) = (f_1(x), 1)$

Triangulating $X \times I$, given a triangulation

$t: |K| \rightarrow X$. Simply do the standard

triangulation of $|K| \times I$ and then use $t \times \text{id}_I$.

Remark: If $\sigma = s(x_0, \dots, x_k)$ is a k -simplex in $\mathbb{R}^N$, we take $s(x_0, \dots, x_k) \times I \subset \mathbb{R}^N \times [0, 1] \subset \mathbb{R}^{N+1}$

The usual triangulation of $\sigma \times I$ depends on the ordering of its vertices; here $x_0 < \dots < x_k$

Then $\Delta \times I$ decomposes into the $(k+1)$ -simplices

$$\tau_i := s((x_0, 0), \dots, (x_i, 0), (x_i, 1), (x_{i+1}, 1), \dots, (x_k, 1)),$$

$$i = 0, \dots, k,$$

together with all faces of the τ_i .

To make the ordering of the vertices natural one first passes to the barycentric subdivision of $K^{(k)}$ and uses the natural order of the vertices of each simplex in $K^{(k)}$; (these are barycenters of a strictly decreasing sequence of faces of some simplex of K).

c.4 Def. (ambient isotopy). $f_0, f_1: X \rightarrow Y$ are called ambient isotopic, if there exists a level preserving p.l. homeomorphism $F: Y \times I \rightarrow Y \times I$ such that

$$F(y, 0) = (y, 0) \text{ and } F(f_0(x), 1) = (f_1(x), 1).$$

Thus F is an isotopy between id_Y and $F_1: Y \rightarrow Y$ given by $F(y, 1) = (F_1(y), 1)$ and $f_1 = F_1 \circ f_0$.

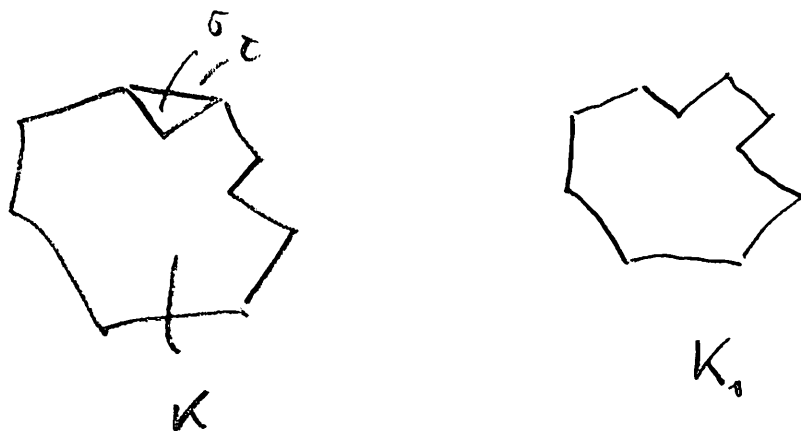
This concept makes sense for any two p.l. maps f_0 and f_1 , not necessarily only for embeddings.

C.5 (Regular Neighborhoods)

Let N be a ^{compact} polyhedron in the p.l. n -manifold M .

A regular neighborhood of N is a p.l. n -submanifold U of M which collapses onto N in the following sense:

Let K be a simplicial complex and $\sigma \in K$ a simplex which contains a face τ s.t. the only simplices of K having τ as a face is τ and σ (then $\dim \tau = \dim \sigma - 1$). Removing σ and τ from K gives us a new complex K_1 and we say that K_1 is obtained from K by an elementary collapse



we say that K collapses to the subcomplex L and write $K \searrow L$ if there is a sequence

$K_0 = K, K_1, K_2, \dots, K_r = L$ s.t. K_i is obtained from K_{i-1} by an elementary collapse.

In the above situation, with $N \subset M$, $U \subset M$ we request that there is a triang. $t: K \rightarrow M$ with subcomplexes L_0, L_1 of K s.t. $t(L_0) = U$, $t(L_1) = N$ and $L_0 \searrow L_1$.

Regular neighborhood always exist:

If K is a simpl. complex, $L \subset K$ a ^{finite} subcomplex

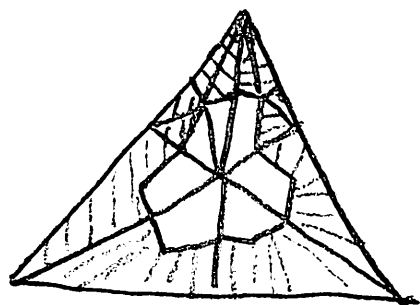
Let $K^{(2)}$ be the second barycentric subdiv. on. Let

$$|U| = \bigcup \{ \sigma \in K^{(2)} : \sigma \cap |L| \neq \emptyset \}.$$

$U = \{ \tau \in K^{(2)} : \exists \sigma \in K^{(2)} \text{ with } \sigma \cap |L| \neq \emptyset \text{ and } \tau \leq \sigma \}$
is the associated subcomplex and $U \supset L^{(2)}$.

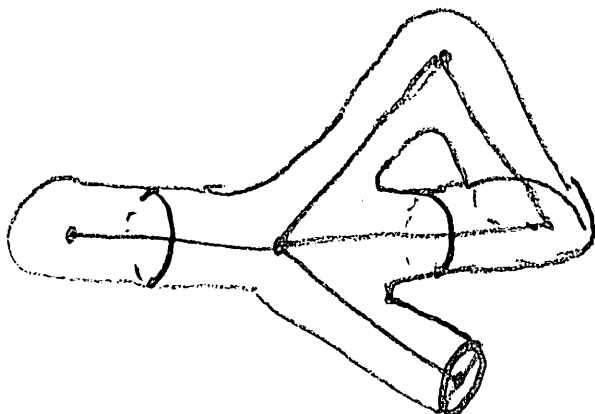
If $|K|$ is an n -manifold, then $|U|$ is a regular neighborhood of $|L|$.

If $|L|$ is the boundary of the 2-simplex $s(x_0, x_1, x_2)$
then $|U|$ is the shaded region



(rounded)

A picture of a regular neighborhood of a graph in $\mathbb{R}^3$ is for example.



The graph may not be in the interior of the regular neighborhood.

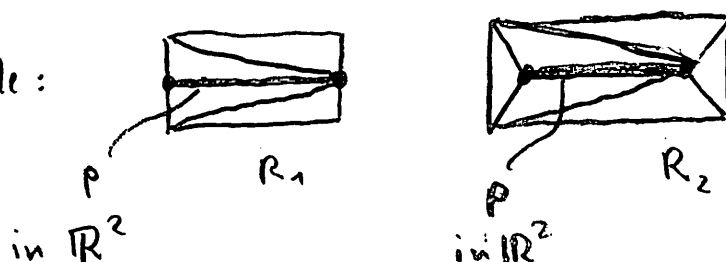
Regular neighborhoods are basically unique.

C.5.1 Theorem: Let P be a polyhedron in the p.l. n -manifold M and R_1, R_2 regular neighborhoods of P .

If $P \subset \text{Int } R_i$, $i=1,2$, then $R_1 \subset M$ and $R_2 \subset M$ are ambient isotopic by an isotopy which keeps P fixed, i.e. $F: M \times I \rightarrow M \times I$ restricted to $P \times I$ is the identity.

In general: If also $R_i \cap \partial M$ is a regular neighborhood of $P \cap \partial M$ then $R_1 \subset M$ and $R_2 \subset M$ are ambient isotopic.

C.5.2 Example:



R_1 and R_2 are ambient isotopic (see below), but when isotoping R_1 into R_2 the image of P after the isotopy has to intersect ∂R_2 in two points, since P intersects ∂R_1 in two points.

C.5.3 Definition: A regular neighborhood of a component C of the boundary ∂M of a p.l. n -manifold M is called a collar of C .

C.5.4 Corollary of C.5.1: Any 2 collars of a component C of ∂M are ambient isotopic by an isotopy fixing C pointwise. Any collar is p.l. homeomorphic to $C \times [0,1]$ with $C = C \times \{0\}$.

To prove the last statement requires some work, similar in nature to prove that for any two complexes K, L in $\mathbb{R}^N$ with $|K| = |L|$ there is a common subdivision K' of K and L .

D. A remark on general position.

This is the analogue in the p.l. category to transversality for smooth manifolds.

The principle behind this is the fact that a k -dimensional and l -dimensional affine subspace of $\mathbb{R}^n$ intersect in a $k+l-n$ dimensional subspace after an arbitrary small change of given spanning set $\{x_0, \dots, x_k\}$ of the first space. (If $k+l < n$ the intersection is $\emptyset$).

One wants to achieve something similar when considering two polyhedra in some triangulated space or self-intersections of p.l. maps.

This can be cumbersome (compare Hempel pp. 9-13)

We do not have to deal with some of the more complicated situations.

For us the following suffices:

D.1 Let S_1, S_2 be p.l. submanifolds ^{of dimension ≤ 2} in a p.l. 3-manifold M . Then there is an arbitrarily small (moving points an arbitrarily small distance) isotopy of M , moving S_1 to S'_1 s.t.

(i) $S'_1 \cap S_2$ is a $\dim S_1 + \dim S_2 - 3$ - manifold
(Remember that the empty set is an n -manifold for all n)

(ii) If $\partial S_1 = \emptyset = \partial S_2$ then $\partial(S_1' \cap S_2) = \emptyset$

(iii) If S_1 and S_2 are proper, i.e. $\partial S_i = S_i \cap \partial M$

then S_1' is proper and $\partial(S_1' \cap S_2) \subset \partial M$

(iii) is only interesting if $\dim S_1 = \dim S_2 = 2$, so that

$$\dim(S_1' \cap S_2) = 1.$$

D. 2 (a) If P_1 and P_2 are 1-dimensional polyhedra in an M^3 then P_1 can be ambiently isotoped into P_1' s.t. $P_1' \cap P_2 = \emptyset$.

(b) If P_1 is 1-dimensional and P_2 is 2-dimensional, then P_1 can be ambiently isotoped to P_1' s.t.

$P_1' \cap P_2$ is a discrete set of points completely contained in $mP_1' \cap mP_2$ where for an i -dim polyhedron P mP are its i -manifold points, i.e. points having a neighborhood in P homeomorphic to $\mathbb{R}^i$.

Our last subject deals with

E. Combinatorial n -balls and their embeddings in p.l. n -manifolds

E. 1 Definition: If B admits a triangulation $t: |K| \rightarrow B$ which $|K|$ p.l. homeomorphic to an n -simplex then B is called a p.l.-(or combinatorial) n -ball

In the same vein we call S a p.l. n-sphere if it admits a triangulation $t: |K| \rightarrow S$ with $|K|$ p.l. homeomorphic to the boundary of an $(n+1)$ -simplex.

Here are some results :

E2. (i) If $B \xrightarrow{f} S$ is a p.l. embedding of a p.l. n-ball into a p.l. n-sphere then

$\overline{S-B}$ is a p.l. n-ball

(ii) Any p.l. n-manifold $|K|$ where K collapses to a point is a p.l. n-ball

(iii) If B is a p.l. n-ball. Then any p.l. homeomorphism $h: \partial B \rightarrow \partial B$ extends to a p.l. homeomorphism of B to B

(iv) Let $B_1^{n-1} \hookrightarrow \partial B_1^n$ and $B_2^{n-1} \hookrightarrow \partial B_2^n$ be p.l. $(n-1)$ -balls in the boundaries of p.l. n-balls B_1^n and B_2^n . Then for any p.l. homeom. $h: B_1^{n-1} \rightarrow B_2^{n-1}$ the space

$$B^n = B_1^n \cup B_2^n, \text{ i.e. } B_1^{n-1} \xrightarrow{h} B_2^{n-1}$$

$$B_1^n \cup B_2^n / \sim \text{ where } \sim \text{ is generated by } x \in B_1^{n-1} \text{ is equiv. to } h(x) \in B_2^{n-1},$$

is again a p.l. n-ball.

Remark: Try to prove this for $n=2$ to see that a proof in general requires some machinery.

E3. Definition: (Orientation of a p.l. n -manifold)

Let $|K| \xrightarrow{t} M$ be a triangulation of a p.l. n -manifold

Then $|K|$ is the union of the n -simplices of K . An $(n-1)$ -simplex is a face of a single n -simplex if it is part of $\partial|K|$ otherwise it is a face of exactly 2 n -simplices.

An orientation of a k -simplex is an ordering of its vertices where 2 orderings define the same orientation if they differ by an even permutation. So for $k \geq 0$ every k -simplex has two orientations.

An orientation of $|K|$ (or of M) is the choice of orientations for the n -simplices of K such that for every $(n-1)$ -simplex τ which is a face of 2 n -simplices σ_1, σ_2 the induced orientations on τ by σ_1 and σ_2 are different.

A manifold is called orientable if it admits an orientation, otherwise it is called non-orientable. If $|K|$ is connected and orientable it has exactly 2 orientations.

A compact ^{connected} p.l. manifold $|K|$ is orientable iff $H_n(|K|, \partial|K|; \mathbb{Z}) \neq 0$. Then $H_n(|K|, \partial|K|; \mathbb{Z}) \cong \mathbb{Z}$, and the 2 generators of $H_n(|K|, \partial|K|; \mathbb{Z})$ correspond to the 2 orientations.

Remark

24

We can relate orientation of simplices to bases of the vector space corresponding to the affine space spanned by the vertices of the simplex:

If $\sigma = S(x_0, \dots, x_k)$ is a k -simplex in $\mathbb{R}^N$ and A_σ is the affine space spanned by σ . For any $a_0 \in A_\sigma$ the set $\{a - a_0 \mid a \in A_\sigma\}$ is a k -dimensional linear subspace V_σ of $\mathbb{R}^N$ independent of the choice of a_0 .

For us $a_0 = \hat{\sigma}$, the barycenter is a good choice because it is fixed under any affine isomorphism induced by a permutation of the vertices (i.e. 0-faces) of σ .

We say that two ^{ordered} bases of a (finite dim.) real vector space V give the same orientation V if the determinant of the base change isomorphism is positive.

In particular, the orientation changes under a permutation of the elements of a basis iff the permutation is odd.

Coming back to σ , an ordering $x_0 < x_1 < \dots < x_k$ of the vertices gives us an ordered basis

$$v_i = x_i - \hat{\sigma}, \quad i = 1, \dots, k$$

of V_σ . A permutation π of the vertices which fixes x_0 induces a permutation $\bar{\pi}$ of the basis and $\text{sign } \pi = \text{sign } \bar{\pi}$. If π is the transposition $x_0 \leftrightarrow x_1$, all other x_i fixed, then the induced linear map of V_σ fixes $v_2, \dots, v_k$ and maps

v_1 to $x_0 - \hat{v} = -(v_1 + v_2 + \dots + v_n)$, so we get the transformation matrix

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ -1 & & 0 & \dots & 1 \end{pmatrix} \text{ of determinant } -1.$$

As a consequence we see that an orientation of an n -simplex corresponds to an orientation of the corresponding subspace of $\mathbb{R}^N$. Thus orientations pass to orientations of the n -simplices of any subdivision.

To see that an orientation of a p.l.- n -manifold M with regard to a triangulation $t: |K| \rightarrow M$ passes to an orientation of M with regard to $t: |K'| \rightarrow M$ for any subdivision, all you need to check is: let σ, τ n -simplices in $\mathbb{R}^n$ inducing the same orientation of $\mathbb{R}^n$ and $\sigma \cap \tau$ is an $(n-1)$ -simplex, then σ and τ induce opposite orientations on $\sigma \cap \tau$.

This is an easy exercise.

After all this elaboration, the following definition makes sense

E.4 (Orientation preserving embeddings).

Let M and N be oriented p.l.-manifolds. A p.l.-embedding $f: M \rightarrow N$ is called orientation preserving if the simplicial embedding $\tilde{f}: |K| \rightarrow |L|$

corresponding to f with regard to some triangulations of M and N map the ^{oriented} n -simplices of K corresponding to the orientation of M orientation preserving to the oriented n -simplices of L corresponding to the orientation of N .

We finally can state our last theorems

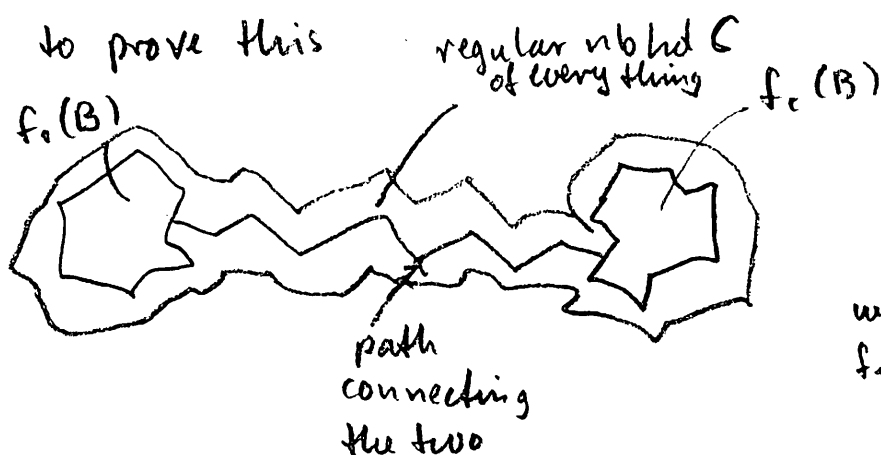
E.5 Any orientation preserving p.l. homeomorphism of a p.l. n -ball or n -sphere is isotopic to the identity

E.6 Let $f_0, f_1 : B^n \longrightarrow M^n$ be p.l. embeddings ^{connected} of a p.l. n -ball into the interior of a p.l. n -manifold M^n .

(a) If M^n is not orientable then f_0 and f_1 are ambient isotopic

(b) If M^n is oriented and f_0 and f_1 both orientation preserving or both orientation reversing, then f_0 and f_1 are ambient isotopic.

Here is a picture suggesting how one might want to prove this



C is a p.l. n -ball, in which we isotope $f_1(B)$ to $f_2(B)$