BACKGROUND INFO SHEET II

This is again just for reference

Basic notions for the homotopy theory of graphs

In what follows we think of a graph as its geometric realization. Thus it is in particular a topological space, but vertices and edges are still part of its structure. We will always think of a graph as being oriented.

We parametrize each edge e by $[0,1]$, i.e. we have a continuous map $f_e : [0,1] \to e \subset \Gamma$ which is an orientation preserving embedding when restricted to $(0,1)$ and maps 0 to $\iota(e)$ and 1 to $\tau(e)$.

1. Proposition. Let Γ be a connected graph, $x_0 \in \Gamma$ (x_0 need not be a vertex) and Y a topological space and $y_0 \in Y$.

(a) To any homomorphism $\varphi : \pi_1(\Gamma, x_0) \to \pi_1(Y, y_0)$ there exists a continuous map $f : (\Gamma, x_0) \to (Y, y_0)$ inducing φ

(b) Let $f_0, f_1 : (\Gamma, x_0) \to (Y, y_0)$ be two cont. maps inducing the same homom. on π_1.
Then there exists a homotopy $f : \Gamma \times I \to Y$ from f_0 to f_1 with $f(\{x_0\} \times I) = y_0$

Proof: (a) should be obvious. First, if x_0 is not a vertex, subdivide the edge containing x_0 into two edges so that x_0 becomes a vertex. This does not change Γ as a topological space.

Now pick a maximal tree T in Γ. For each edge e not in T let $w_e = [x_0, \iota(e)]_T \, e \, [\tau(e), x_0]$, considered as a loop in Γ. Then $\{[w_e] : e \text{ not in } T\}$ is a basis of the free gp. $\pi_1(\Gamma, x_0)$. Let $v_e : [0,1] \to Y$ be a representing loop of $\varphi([w_e]) \in \pi_1(Y, y_0)$. Then define $f : (\Gamma, x_0) \to (Y, y_0)$ as follows:

$f(T) = \{y_0\}$. $\quad f|_e = v_e \qquad$ i.e.

if $f_e : [0,1] \to e$ parametrizes e then

$$f(f_e(t)) = v_e(t), \quad 0 \le t \le 1.$$

Then $f_*([w_e]) = [v_e] = \varphi([w_e])$. Since $\{[w_e] : e \text{ not in } T\}$ is a basis $f_* = \varphi$.

(b) Not much harder than the proof of (a). Again we may assume that $x_0 \in V_\Gamma$. Then

$$\Gamma \times I = (V_\Gamma \times I) \cup (\text{edges in } T) \times I \cup (\text{edges not in } T) \times I.$$

We construct the homotopy f from f_0 to f_1 starting on $V_\Gamma \times I = \{x_0\} \times I \cup (V_\Gamma \setminus \{x_0\}) \times I$

mapping $\{x_0\} \times I$ to y_0 and for $v \in V_\Gamma \setminus \{x_0\}$ we map $v \times I$ to the path

$$f_0 \circ [v, x_0]_T * f_1 \circ [x_0, v]_T \qquad \text{i.e}$$

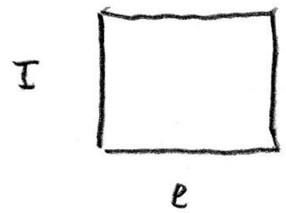
if $w_v : [0,1] \to T$ parametrizes the path $[x_0, v]_T$

then $(v,t) \in v \times T$ gets mapped (by f) to

$$\begin{cases} f_0(w_v(1-2t)) & 0 \le t \le \frac{1}{2} \\ f_1(w_v(2t-1)) & \frac{1}{2} \le t \le 1 \end{cases}$$

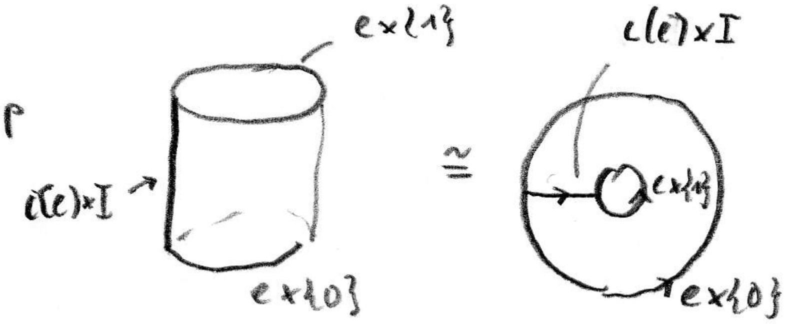
In particular, $f(v,0) = f_0(v)$, $f(v,\frac{1}{2}) = x_0$, $f(v,1) = f_1(v)$.

Now let e be an edge of T; look at $e \times I$



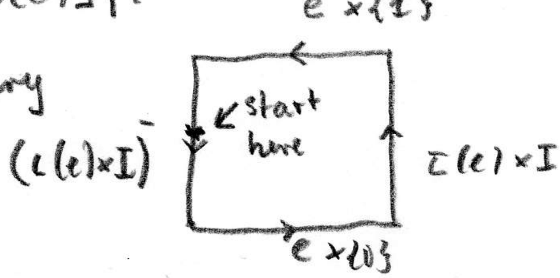
then f is already defined on the boundary of the square or

in case of a loop



The complement is in both cases the interior of a square. For simplicity assume that e is oriented in such a way that e is part of $[x_0, \tau(e)]_T$ (and thus e is not part of $[x_0, \iota(e)]_T$.

Then the boundary



of the square is mapped to

$$f_0 \circ [x_0, \iota(e)]_T * f_0(e) * f_0 \circ [\tau(e), x_0]_T * f_1 \circ [x_0, \tau(e)]_T \circ f_1(e)^{-} * f_1 [\iota(e), x_0]_T$$