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Free Groups and Graphs

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Homework 6 Due: November 26, 2012

Problem 1

Let S be a subgroup of the free group $F(T)$ on the alphabet T, and let $R_S = \{w_\alpha \mid \alpha \in A\}$ be a set of reduced words in $T \sqcup T^{-1}$ such that $\{[w_\alpha] \mid \alpha \in A\}$ forms a system of representatives for the set of right cosets

$$S \backslash F(T) = \{Sg \mid g \in F(T)\}$$

of S in $F(T)$. Call R_S a *Schreier system* if whenever we have a factorization $w_\alpha = w_1 \cdot w_2$ then the word w_1 also belongs to R_S.

Show, making use of the theory of coverings of graphs, that for *every* subgroup S of $F(T)$ there is a Schreier system of representatives for $S\backslash F(T)$.

Hint: Let $G(T)$ be the graph with $V(G(T)) = \{v\}$ and $\pi_1(G(T), v) \cong F(T)$, and consider the covering graph of $G(T)$ corresponding to the subgroup $S \leq F(T)$.

Problem 2

Let S be a subgroup of $F(T)$ and $R_S = \{w_\alpha \mid \alpha \in A\}$ a Schreier system for $S\backslash F(T)$. Consider the set $E_S = \left\{[w_\alpha \cdot t] \cdot \left[\overline{w_\alpha \cdot t}\right]^{-1} \mid \alpha \in A, t \in T\right\} \subseteq F(T)$, where $\overline{w_\alpha \cdot t}$ is the word in R_S such that

$$S\left[\overline{w_\alpha \cdot t}\right] = S\left[w_\alpha\right]\left[t\right].$$

Show that $E_S \setminus \left\{1_{F(T)}\right\}$ forms a basis of S.

Hint: You may solve this problem either way you like. However, making use of the theory of graphs as in problem 1 might simplify things.