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Free Groups and Graphs

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Homework 5 Due: November 19, 2012

In this homework we complete the proof that the fundamental group of a pointed combinatorial graph is the "same" as the fundamental group of its associated pointed topological space.

Problem 1

Denote the category of pointed topological spaces (objects are spaces together with a basepoint and morphisms are basepoint-preserving continuous maps) by $\mathbf{Top}_\bullet$ and by $\pi_1 \colon \mathbf{Top}_\bullet \to \mathbf{Grp}$ the fundamental group functor. Denote the *combinatorial* fundamental group functor from the category of pointed graphs $\mathbf{Graphs}_\bullet$ to groups $\mathbf{Grp}$ as defined in the lecture by π_1^{comb}. Also as in the lecture, denote the geometric realization of a graph Γ by $G(\Gamma)$.

(*i*) Define for each graph homomorphism $f \colon \Gamma \to \Delta$ a continuous map of geometric realizations $G(f) \colon G(\Gamma) \to G(\Delta)$ such that $G \colon \mathbf{Graphs} \to \mathbf{Top}$ is a functor.

(*ii*) Show that there is a natural transformation $\psi \colon \pi_1^{\text{comb}} \to \pi_1 \circ G$ such that for every pointed graph (Γ, v) the homomorphism

$$\psi_{(\Gamma,v)} \colon \pi_1^{\text{comb}}(\Gamma, v) \to \pi_1(G(\Gamma), v)$$

is surjective. (Hint: Look at homework sheets 3 and 4.)

Problem 2

Homotopies of paths involve maps from the unit square into spaces. So we have to leave the one-dimensional world of graphs and their geometric realizations briefly and look at 2-dimensional simplicial complexes. Most of you are probably familiar with the following notions and their generalizations to all dimensions.

(*i*) A 0-simplex in $\mathbb{R}^n$ is a point, a 1-simplex in $\mathbb{R}^n$ is a compact segment of a straight line of finite positive length, a 2-simplex in $\mathbb{R}^n$ is a nondegenerate solid triangle.

(*ii*) 0-simplices have no boundary simplices, the boundary simplices of a 1-simplex are its two end points, and the boundary simplices of a 2-simplex are its three edges (1-simplices) and its three vertices (0-simplices). An *open* simplex is a simplex minus the points in its boundary simplices.

(*iii*) A 2-dimensional simplicial complex in $\mathbb{R}^n$ is a set K of 0-, 1-, and 2-simplices in $\mathbb{R}^n$ such that

- if σ is a simplex in K, all boundary simplices of σ are also in K;
- any two distinct simplices of K are either disjoint or they intersect in a common boundary simplex.

(*iv*) If K is a simplicial complex we denote by $|K|$ the union of its simplices considered as a subspace of $\mathbb{R}^n$.

(*v*) The (open) star $st(v, K)$ of the 0-simplex v of the simplicial complex K is the union of v and all open simplices whose associated simplex contains v as a boundary simplex.

For each $n > 0$ consider the 2-dimensional simplicial complex $K(n)$ in $\mathbb{R}^2$ with $|K(n)| = [0,1]^2$ that consists of the 2-dimensional simplices whose vertices are $\{(\frac{i}{n}, \frac{j}{n}), (\frac{i+1}{n}, \frac{j}{n}), (\frac{i+1}{n}, \frac{j+1}{n})\} \cup \{(\frac{i}{n}, \frac{j}{n}), (\frac{i}{n}, \frac{j+1}{n}), (\frac{i+1}{n}, \frac{j+1}{n})\}, 0 \leq i, j \leq n-1$, and all their boundary simplices.

Let Γ be a graph and $\Gamma^{(2)}$ its second subdivision. Let $h\colon [0,1]^2 \to G(\Gamma^{(2)})$ be a linearly parametrized simplicial map (i.e. simplices go to simplices) which maps the left, right and top edge of the square $[0,1]^2$ to the basepoint vertex v of Γ. Show that the edge path associated to the map h restricted to $[0,1] \times \{0\}$ can be reduced to the constant path (as a path in a graph).

Problem 3

More generally, the previous result holds for continuous (and not necessarily simplicial) maps $h\colon [0,1]^2 \to G(\Gamma^{(2)})$ that map each point $(\frac{i}{n}, 0)$, $1 \leq i \leq n-1$, to a vertex of $G(\Gamma^{(2)})$, assuming that for every 0-simplex w of $K(n)$ there exists a vertex $v(w)$ of $G(\Gamma^{(2)})$ such that $h(st(w, K))$ is contained in the star of $v(w)$ in $\Gamma^{(2)}$.

Use this fact to show that for every pointed graph (Γ, v) the surjective homomorphism $\psi_{(\Gamma,v)}$ from Problem 1 is in fact an isomorphism.