Freie Universität Berlin
Institut für Mathematik

Prof. Dr. Elmar Vogt
Sebastian Meinert

Free Groups and Graphs

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Homework 3 Due: November 5, 2012

Problem 1

For a group G the *commutator subgroup* $[G, G]$ is defined as the subgroup of G generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$. Show that $[G, G]$ is the smallest normal subgroup H of G such that G/H is abelian. Furthermore, show that $F_n/[F_n, F_n]$ is isomorphic to the free abelian group $\mathbb{Z}^n$.

Problem 2

Recall that a sequence

$$\dots \stackrel{f_{i+1}}{\to} G_i \stackrel{f_i}{\to} G_{i-1} \stackrel{f_{i-1}}{\to} \dots$$

of homomorphisms between groups is called *exact* if for all i the equation $im(f_i) = ker(f_{i-1})$ holds. An exact sequence of the form

$$1 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 1$$

is called a *short exact sequence*. Such a sequence is said to *split* if there exists a homomorphism $s : C \to B$ with $\beta \circ s = id_C$. Such an s is called a *splitting* of the short exact sequence.

(a) Let $1 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 1$ be a split short exact sequence. Show that then there exists a homomorphism $\varphi : C \to Aut(A)$ such that B is isomorphic to the group with underlying set $A \times C$ and multiplication given by

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 \cdot \varphi(c_1)(a_2), c_1 \cdot c_2).$$

For this reason one calls the group B in a split short exact sequence $1 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 1$ a *semidirect product* of A and C.

(b) Denote by R_n the graph with 1 vertex and $2n$ (oriented) edges and by $\mathcal{G}_n$ the group of graph automorphisms of R_n. Show that for all n there exists a nontrivial split short exact sequence

$$1 \to N \to \mathcal{G}_n \to Q \to 1$$

with an abelian group N.

Problem 3

A *subdivision* of a graph Γ is a graph Γ' with the following properties:

- $V_\Gamma \subset V_{\Gamma'}$
- for each edge e of Γ there is a reduced path $p_e = a_1 \cdots a_{n_e}$ in Γ' from ιe to τe with the vertices $\iota a_1, \iota a_2, \ldots, \iota a_{n_e}$ pairwise distinct
- if $p_e = a_1 \cdots a_{n_e}$ then $p_{\bar{e}} = \overline{a_{n_e}} \cdots \overline{a_1}$
- $E_{\Gamma'} = \bigsqcup_{e \in E_\Gamma} \{\text{edges of } p_e\}$, where by $\bigsqcup$ we mean the *disjoint* union

Geometrically, one obtains Γ' from Γ by replacing each geometric edge $\{e, \bar{e}\}$ with a path of length n_e, adding in the process $n_e - 1$ new vertices in the interior of the edge.

Let Γ' be a subdivision of Γ. Show that for every vertex v of Γ there is an isomorphism $\pi_1(\Gamma, v) \to \pi_1(\Gamma', v)$.

Problem 4

For a graph Γ denote by $\Gamma^{(n)}$ the subdivision of Γ (refer to the previous problem for notation) where $n_e = n + 1$ for every edge e of Γ. In particular $\Gamma^{(0)}$ is canonically isomorphic to Γ. Show that there is functor $S^{(n)}$ from the category of graphs to itself with $S^{(n)}(\Gamma) = \Gamma^{(n)}$ and for any vertex v of a graph Γ an isomorphism $\pi_1(\Gamma, v) \xrightarrow{\varphi_\Gamma} \pi_1(\Gamma^{(n)}, v)$ which is natural in the sense that for any graph map $f : \Gamma \to \Delta$ the following diagram commutes:

$$\begin{array}{ccc} \pi_1(\Gamma, v) & \xrightarrow{\varphi_\Gamma} & \pi_1(\Gamma^{(n)}, v) \\ \downarrow{\scriptstyle \pi_1(f)} & & \downarrow{\scriptstyle \pi_1(S^{(n)}(f))} \\ \pi_1(\Delta, f(v)) & \xrightarrow{\varphi_\Delta} & \pi_1(\Delta^{(n)}, f(v)) \end{array}$$