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Free Groups and Graphs

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Homework 2 Due: October 29, 2012

Problem 1

The *center* of a group G is the subgroup $Z(G) = \{z \in G \mid gz = zg \text{ for all } g \in G\}$. Compute the center of F_n, the free group of rank n, for all $n \in \mathbb{N}$.

Problem 2

Show that an index 2 subgroup of any group is normal. Use this to show that F_2 has exactly 3 subgroups of index 2. If you are eager to go on, show that F_n, $n \geq 2$, has exactly $2^n - 1$ subgroups of index 2.

(Hint: If $N \leq G$ is normal, G/N is a group and we get a surjective group homomorphism $G \to G/N$.)

Problem 3

A *group action* of a group G on a set X is a group homomorphism $\phi : G \to Aut(X)$, where $Aut(X)$ denotes the group of automorphisms (or permutations) of X. For example, the additive group of integers $\mathbb{Z}$ acts on the real line $\mathbb{R}$ via the group action $k \mapsto \tau_k$, where τ_k denotes the automorphism of $\mathbb{R}$ given by $x \mapsto x + k$. In the field of *geometric group theory* one studies algebraic properties of groups by studying their actions on certain sets, or more specifically topological spaces. If G acts on X, for $g \in G$ and $x \in X$ we denote $(\phi(g))(x) \in X$ simply by gx.

Let G be a group acting on a set X and $a, b \in G$ elements of infinite order. Assume there exist two nonempty disjoint subsets A, B of X such that $a^k B \subseteq A$ and $b^k A \subseteq B$ for all $k \in \mathbb{Z} \setminus \{0\}$. Show that the subgroup of G generated by a and b is a free group of rank 2.

Problem 4

Show that the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free subgroup of $SL_2(\mathbb{Z})$.