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Free Groups and Graphs

Winter 2012/2013

Homework 1 Due: October 22, 2012

The tutorials will take place on *Mondays*, 4 - 6 pm, at Arnimallee 6, SR 032. The first tutorial will take place on Monday, October 22, 2012.

Problem 1

Show that the reduction-of-words algorithm from the lecture reduces two equivalent words to the same reduced word. This proves that if two reduced words are equivalent then they are in fact equal.

Problem 2

Let T be a set and $w = t_1^{\epsilon_1} \cdots t_n^{\epsilon_n} \in W(T)$ with $t_i \in T,\ \epsilon_i \in \{\pm 1\}$ a word in $T \sqcup T^{-1}$. Say that w is *cyclically reduced* if it is reduced and $t_n^{\epsilon_n} \neq t_1^{-\epsilon_1}$, i.e. the last letter of w is not the inverse of the first letter. Denote the cyclic reduction of w by $\mathrm{cr}(w)$, and denote by $[w]$ the class of w in $F(T)$, the free group with basis T. Furthermore, call any word of the form $t_r^{\epsilon_r} \cdots t_n^{\epsilon_n} \cdot t_1^{\epsilon_1} \cdots t_{r-1}^{\epsilon_{r-1}}$ a *cyclic permutation* of w.

Given $w_1, w_2 \in W(T)$, show that $[w_1]$ and $[w_2]$ are conjugate if and only if $\mathrm{cr}(w_1)$ and $\mathrm{cr}(w_2)$ are cyclic permutations of each other.

Problem 3

This problem might be a little bit tricky. However, please don't be discouraged! Spend some time on this in order to develop a sense for words and their reduced representations.

Show that if $x, y \in F(T)$ are commuting elements, i.e. $xy = yx$, then there exist $z \in F(T),\ k, l \in \mathbb{Z}$ such that $x = z^k$ and $y = z^l$.

(Hint: Let u and v be reduced representatives of x and y respectively. Denote by $|u|$ the length of u (the number of letters after reduction). One may assume w.l.o.g. that $|u| \leq |v|$ and u is cyclically reduced.)

Problem 4

A free group is of *rank* n if it has a basis of cardinality n. As all free groups of rank n are isomorphic, we may speak of *the* free group of rank n, denoted by F_n. One easily sees that F_n contains a free subgroup of rank k for all $k \leq n$. More surprisingly, one can also show that for $n \geq 2$ the group F_n contains a free subgroup of *every* finite and even of countably infinite rank.

Show that $\left\{b^k a b^{-k} \mid k \in \mathbb{Z}\right\}$ forms a basis of a free subgroup of $F_2 \cong F(\{a, b\})$ of countably infinite rank.