The next application to group theory depends (among other things) on:

4.10 Proposition: Let Δ be a graph with one vertex, Γ a graph with finitely many vertices and let $f : \Gamma \to \Delta$ be an immersion. Then there exists a graph Γ' with $\Gamma \subset \Gamma'$ and $\Gamma' \setminus \Gamma$ consisting only of edges and a graph map f' extending f with f' a covering. (If Δ is finite the proof shows how to construct effectively all such extensions (Γ', f') up to isomorphism)

Proof. Choose an oriention $\mathcal{V}$ for Δ and define for each $e \in \mathcal{V}$

$$R_e = \{(u,v) \in V_\Gamma \times V_\Gamma : \text{there exists } e_1 \in E_\Gamma \text{ with } f(e_1) = e, \ \iota(e_1) = u, \ \tau(e_1) = v\}$$

if $(u, v_1), (u, v_2) \in R_e$ then the corresponding edges e_1, e_2 of $st(u, \Gamma)$ get mapped to e. Since f is an immersion we have $e_1 = e_2 \Rightarrow v_1 = v_2$. Similarly, $(u_1, v), (u_2, v)$ both in R_e implies $u_1 = u_2$. Therefore, R_e defines a bijective map of

$$\{u \in V_\Gamma : \exists v \text{ with } (u,v) \in R_e\} \text{ to } \{v \in V_\Gamma : \exists u \text{ with } (u,v) \in R_e\}$$

Since V_Γ is finite this map can be extended to a bijection S_e of V_Γ to V_Γ. Choose such an S_e for all $e \in \mathcal{V}$ and define Γ', f' as follows

$$V_{\Gamma'} = V_\Gamma$$

$$E_{\Gamma'} = \{(u,v,e) \in V_\Gamma \times V_\Gamma \times E_\Delta : \text{if } e \in \mathcal{V} \text{ then } (u,v) \in S_e, \text{ if } \bar{e} \in \mathcal{V} \text{ then } (v,u) \in S_{\bar{e}}\}$$

$\overline{(u,v,e)} = (v,u,\bar{e})$ (then clearly $(u,v,e) \neq \overline{(u,v,e)}$ and $\overline{\overline{(u,v,e)}} = (u,v,e)$)

$\iota(u,v,e) = u$

And so Γ' is a graph; define

$f' = f$ on vertices and

$f'(u,v,e) = e$.

$\Gamma < \Gamma'$ for vertices is clear, and

$e_1 \in E_\Gamma$ corresponds to $(\iota(e_1), \tau(e_1), f(e_1))$.

It remains to show that f' is a covering. For $u \in V_\Gamma = V_{\Gamma'}$ we have

$$St(u,\Gamma') = \{(u,v,e) : (u,v,e) \in E_{\Gamma'}\}$$

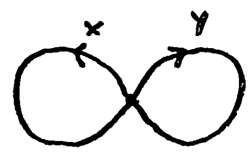
Since S_e for $e \in \mathcal{V}$ is a bijection $V_\Gamma \longrightarrow V_\Gamma$ for any $e \in E_\Delta = \mathcal{V} \sqcup \bar{\mathcal{V}}$ there exists a unique v s.t. $(u,v,e) \in E_{\Gamma'}$. This shows that

$f'|_{St(u,\Gamma')} : St(u,\Gamma') \longrightarrow E_\Delta$ is a bijection. $\square$

To see this process at work in a particular example.

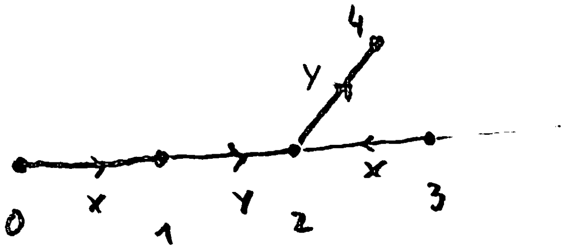
<u>4.11 Example</u> (To be supplemented by other examples of your choice; one you find in Stallings paper)
Let F be free on generators $\{x,y\}$. We want to determine all subgroups S of index 5 having $\{1, x, xy, xyx^{-1}, xy^2\}$ as representatives of right cosets (i.e. cosets of the form $S\cdot g$)

Represent F as $\pi_1(\Delta)$ with

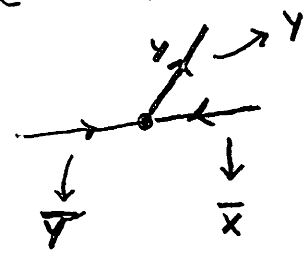
$\Delta =$ 

the classes of oriented loops representing the generator of F given by the label.

Consider Γ as the tree below



where the map into Δ is indicated by the arrows and labels. This is an immersion (st$(2, \Gamma)$



)

R_x : $0 \longrightarrow 1$
$3 \longrightarrow 2$

R_y : $1 \longrightarrow 2 \longrightarrow 3$

Each one can be extended in $3 \cdot 2$ ways to a bijection of $\{0,1,2,3,4\}$. So there are 36 ways to extend this immersion to covering without adding vertices.

<u>Consequence</u>: There are exactly 36 subgroups S of $F\{x,y\}$ having $1, x, xy, xy^2, xyx^{-1}$ as coset representatives.

(we have actually only shown: $\exists$ at most 36)

4.12 Corollary (M. Hall jr (1949), R.G. Burns 1969)

Let $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$ be elements of a free gp. F. Let S be the subgp. gen. by $\{\alpha_1, \dots, \alpha_m\}$ and assume that $\beta_i \notin S$ for all i.

Then there exists a subgroup S' of F of finite index with $S \subset S'$ and $\beta_i \notin S'$ for all i. Furthermore S' can be chosen so that it has a basis a subset of which is a basis of S.

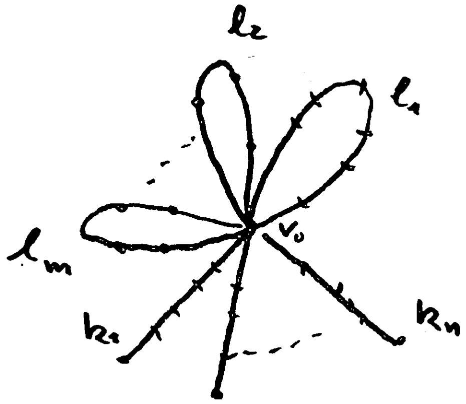
Proof. Write α_i as reduced words w_i

β_i u_i

Take Γ_1 to be m simple loops $l_1, \dots, l_m$ with each loop having exactly $|w_i|$ vertices

and n paths $k_1, \dots, k_n$

each path having exactly $|u_i|$ edges (and $|u_i|+1$ vertices) and identify all initial vertices to a single vertex v_0



and map $\Gamma_1 \xrightarrow{f_1} \Delta$ as indicated by

$w_1, \dots, w_m, u_1, \dots, u_k$

Then clearly,

$$f_1(\pi_1(\Gamma_1, v_0)) = S$$

By 3.5 f_1 can be written in the form $f \circ p_r \circ \cdots \circ p_1$ where each p_i is a fold and f an immersion $f : \Gamma \longrightarrow \Delta$

By 3.16a p_i are surjective on π_1 so that

$f\pi_1(\Gamma, v_0)$ is still equal to S.

Since $\beta_i \notin S$ the image in Γ of any of the paths corresponding to $k_1, \dots, k_n$ are not loops.

Now we apply 4.10, extending $f : \Gamma \longrightarrow \Delta$ to a covering $f' : \Gamma' \longrightarrow \Delta$ of index $= \# V_\Gamma$ which is finite, $f' : \pi_1(\Gamma', v_0) \longrightarrow S' \subset \pi_1(\Delta, *)$ is a subgroup of index $\# V_\Gamma$, the images of $k_1, \dots, k_n$ in Γ' are those in Γ and hence not closed.

A tree for Γ is a tree for Γ', so the corresponding basis for $\pi_1(\Gamma', v_0)$ contains the corresponding basis for $\pi_1(\Gamma, v_0)$. This proves the corollary. $\square$