Then, clearly, $\overline{[p]\cdot\tilde{e}} := [p]\tilde{\bar{e}} \neq [p]\cdot\tilde{\bar{e}}$, so that $(\tilde{\Gamma}, \tilde{v})$ satisfies one crucial condition for being a G-graph, $G = \pi_1(\Gamma, v)$. Furthermore: $[1]\tilde{w} = \tilde{w}$, $[1]\tilde{e} = \tilde{e}$ for all vertices $\tilde{w}$ and edges $\tilde{e}$ of $\tilde{\Gamma}$; if $[p_1], [p_2] \in \pi_1(\Gamma, v)$ and $\tilde{q}$ a path from $\tilde{v}$ to $\tilde{w}$ in $\tilde{\Gamma}$ then

$$[p_1]([p_2]\tilde{w}) = [p_1](\tau(\widetilde{p_2 q})) = \tau(\widetilde{p_1 p_2 q})$$

$$= ([p_1][p_2])\tilde{w}$$

where, as before, $q = f\tilde{q}$.

Thus we have a $\pi_1(\Gamma, v)$ action.

<u>It is free</u>: since

$[p]\tilde{w} = \tilde{w}$ implies $\tau(\widetilde{pq}) = \tilde{w}$, so that $\widetilde{pq}$ and $\tilde{q}$ are paths from $\tilde{v}$ to $\tilde{w}$. Then $\widetilde{pq}\,\bar{\tilde{q}}$ is a closed path in $\tilde{\Gamma}$ and thus homotopic to the constant path. Thus $pq\cdot\bar{q}$ is homotopic to c_v i.e. $p \simeq c_v$ i.e. $[p] = 1_{\pi_1(\Gamma, v)}$.

<u>3.14 Corollary to 3.9 + 2.18</u> Subgps of free gps are free.

<u>Pf</u>: Any free gp F is (isom. to) the fundamental group of some graph.

Any subgp. $H \leq F$ is the fundamental gp. of a covering graph. Therefore (2.18) H is free.

3.15 Remarks.

Let $f:(\tilde{\Gamma},\tilde{v})\to(\Gamma,v)$ be the universal cover of the conn. graph Γ. Then $G=\pi_1(\Gamma,v)$ acts freely on $\tilde{\Gamma}$ and $\tilde{\Gamma}/G\cong\Gamma$. If H is any subgroup of G then $\tilde{\Gamma}/H$ is a covering of Γ with $\pi_1(\tilde{\Gamma}/H, H\cdot\tilde{v})\cong H$ canonically. So

$$\begin{aligned}\tilde{\Gamma}/H &\longrightarrow \Gamma\\ H\tilde{e} &\longmapsto f(\tilde{e})\\ H\cdot\tilde{w} &\longmapsto f(\tilde{w})\end{aligned}$$

is *the* covering associated to H.

Here is an easy application of covering theory for graphs.

Define the *join* $A\vee B$ *of two subgps* of a gp G as the subgroup generated by $A\vee B$. Then we have

3.16 Proposition:

Let
$$\begin{tikzcd} \Gamma \arrow[r, "\alpha_1"] \arrow[d, "\alpha_2"'] & \Delta_1 \arrow[d, "\beta_1"] \\ \Delta_2 \arrow[r, "\beta_2"'] & \Delta \end{tikzcd}$$

be a push-out diagram with $\Gamma, \Delta_1, \Delta_2$ connected. Let $u \in V_\Gamma$, $v_i \in V_{\Delta_i}$ be vertices with $v_i = \alpha_i(u)$ and denote $\beta_i(v_i)$ by w ($\beta_1 v_1 = \beta_2 v_2$, so this is well-defined)

As usual, we denote the map induced by β_i on the fundamental groups again by β_i. Then

$$(*) \qquad \pi_1(\Delta, w) = \beta_1 \pi_1(\Delta_1, v_1) \vee \beta_2 \pi_1(\Delta_2, v_2)$$

Proof. Denote the subgroup of $\pi_1(\Delta, w)$ given by the right hand side of $(*)$ by S and let $(\tilde{\Delta}, \tilde{w}) \xrightarrow{f} (\Delta, w)$ be the covering given by S (exists by 3.9.(2)).

By 3.8 there exist unique lifts $(\Delta_i, v_i) \xrightarrow{\tilde{\beta}_i} (\tilde{\Delta}, \tilde{w})$ of β_i, $i = 1, 2$, and $\tilde{\beta}_i \alpha_i$ are lifts of $\beta_i \alpha_i$. By uniqueness of lifts we have $\tilde{\beta}_1 \alpha_1 = \tilde{\beta}_2 \alpha_2$. The push-out property gives us a unique $g : \Delta \longrightarrow \tilde{\Delta}$ s.t.

$g \beta_i = \tilde{\beta}_i$. Then $f \circ g \circ \beta_i = f \circ \tilde{\beta}_i = \beta_i$. Again by the push-out property $f \circ g = id_\Delta$. Therefore f is a surjective map on π_1. Since it is also injective it is an isomorphism. $\square$

3.16a Mini Corollary: If (e_1, e_2) is an admissible edge pair in Γ then $\Gamma \longrightarrow \Gamma / [e_1 = e_2]$ is surj. on π_1. $\square$
(Look at page 3.4 for the relevant diagram)

Next we deal with some properties of immersions.

3.17 Proposition: Let $f : \Gamma \to \Delta$ be an immersion

(a) If p is a reduced path in Γ then fp is also reduced

(b) If p, q are paths in Γ with $\iota p = \iota q$ and $fp = fq$ then $p = q$

(c) If Θ is a connected graph and $g_1, g_2 : \Theta \to \Gamma$ are two maps with $fg_1 = fg_2$ and $g_1(w) = g_2(w)$ for one vertex w of Θ. Then $g_1 = g_2$

Proofs of (a) and (b) are left as easy exercises. (c) follows from (b) since Θ is connected.

3.18 Proposition: (Injectivity of π_1) If $f : \Gamma \to \Delta$ is an immersion and $v \in V_\Gamma$ then

$$f : \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$$

is injective.

Proof: Let p be a reduced loop starting in v with $[p] \neq 1$ in $\pi_1(\Gamma, v)$. By 3.17(a) fp is reduced and of length $|p| > 0$. By (2.14) each homotopy class of paths in any graph contains a unique reduced path. Therefore $[fp] \neq [c_{f(v)}]$. $\square$

So we see that immersions also give us injective homomorphisms onto subgroups, and these tend to be more effective, since we have smaller graphs in comparison to coverings (in general; by definition a covering is also an immersion).

Here is an algorithm which produces a "small" immersion for <u>finitely generated</u> subgroups

<u>3.19 Proposition</u> Given a finite set $S = \{x_1, \dots, x_n\} \subset \pi_1(\Delta, w)$ there is an algorithm that represents the subgp. H generated by S by an immersion $f: \Gamma \to \Delta$

<u>Description</u>: Represent x_i by a loop p_i starting at w. Let $B_1, \dots, B_n$ be standard arcs of length $|p_1|, |p_2|, \dots, |p_n|$, and let Γ_1 be their disjoint union. Map Γ_1 to Δ by $p_1 \cup p_2 \cup \dots \cup p_n$. Identifying in Γ_1 all $\iota(B_i)$, $\tau(B_i)$ to a single point v we obtain Γ_2 and a map $f_2 : \Gamma_2 \to \Delta$. Then $f_2(\pi_1(\Gamma_2, v_2)) = H$

By 3.5 there is a sequence of foldings

$$\Gamma_2 \to \Gamma_3 \to \dots \to \Gamma_r \quad \text{and a unique immersion}$$

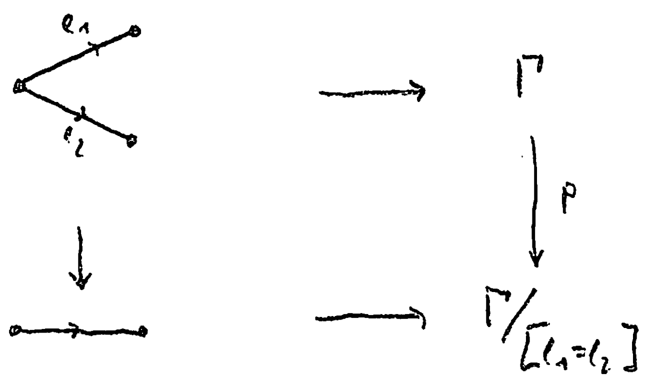
$$f: \Gamma_r \longrightarrow \Delta \quad \text{with } f_2 \text{ being the composition.}$$

Since foldings are surjective on fundamental groups

$$f(\pi_1(\Gamma_r, v_r)) = H \quad \text{where } v_r \text{ is the image of } v_2 \text{ in } \Gamma_r.$$

<u>3.20</u> It is easy to see, when a folding induces an isom. in π_1. This will be helpful to see whether $S = \{x_1, \dots, x_n\}$ is in fact a basis of H.

Recall 3.16a



If $\tau(e_1)\neq\tau(e_1)$ then p is an iso. on π_1

If $e_1\neq e_2$ and $\tau(e_1)=\tau(e_2)$ then p kills one basis element of some basis of $\pi_1(\Gamma)$.

In the first case consider a tree for Γ constructed as in 2.17 with base vertex $\iota(e_1)=\iota(e_2)$, if none of e_1 or e_2 is a loop we may choose e_1, e_2 to be in the tree; otherwise, let e_1 be a loop (then e_2 is not a loop) then we may choose e_2 to be in the tree.

In the first case, we may choose $e_1=e_2$ to be in the tree for $\Gamma/_{[e_1=e_2]}$, the tree being the image of the tree in Γ.
In the second, $e_1=e_2$ is a loop in $\Gamma/_{[e_1=e_2]}$ which is the image of the loop e_1 in Γ. In either case p maps the corresponding bases of $\pi_1(\Gamma)$ to the basis of $\pi_1(\Gamma/_{[e_1=e_2]})$ where you may take any basepoint in Γ.
The tree of $\Gamma/_{[e_1=e_2]}$ is the image of the tree in Γ after removal of the image of e_2 (which is a loop).

If $\tau(e_1)=\tau(e_2)$ and e_1 is not a loop, we may take e_1 in the tree and e_2 corr. to a basis element. This basis elt. is killed by p.
If e_1 (and thus also e_2) are loops they represent different elements of the basis which are identified by p.

$\square$